ON THE PALINDROMIC AND PRIMITIVE WIDTHS OF A FREE GROUP

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Introduction

Let G be a group and $S \subseteq G$ a subset that generates G. For each $x \in G$ define the *length* $l_S(x)$ of x relative to S to be the minimal k such that x is a product of k elements of S. The supremum of the values $l_S(x)$, $x \in G$, is called the *width* of G with respect to S and is denoted by $\mathrm{wid}(G, S)$. In particular, $\mathrm{wid}(G, S)$ is either a natural number or ∞. If $\mathrm{wid}(G, S)$ is a natural number, then every element of G is a product of at most $\mathrm{wid}(G, S)$ elements of S.

Many group-theoretic results can be interpreted as determining the width of a group with respect to one generating set or another. We can mention results on the width of matrix groups with respect to the set of transvections (see e.g. [1, 4, 20]) and the study of the width of verbal subgroups of various free constructions ([2, 7, 8, 17]).

It is also worth mentioning that the concept of the width of a group can be useful in model theory. Possible applications here are based on the following simple argument. Consider a first-order definable subset D of a group G; in other words, D is the set of realizations of a first-order formula in the language of groups. Then the subgroup $\langle D \rangle$ generated by D is also definable in G, provided that the width of $\langle D \rangle$ relative to D is finite. For example, one of the authors proved that the family of all inner automorphisms determined by powers of primitive elements is definable in the automorphism group $\mathrm{Aut}(F_n)$ of the free group F_n of rank $n \geqslant 2$ [19]. It is however an open question whether or not the subgroup $\mathrm{Inn}(F_n)$ of all inner automorphisms is definable in $\mathrm{Aut}(F_n)$. An affirmative answer would follow from the finiteness of the width of F_n relative to the set of all primitive elements (the *primitive width* of F_n.) Similarly, inner automorphisms from $\mathrm{Aut}(F_n)$ determined by palindromic words in F_n with respect to a fixed basis of F_n are also definable – this time with suitable definable parameters – in $\mathrm{Aut}(F_n)$. Palindromic words generate F_n, and this again raises the question of whether or not the corresponding (*palindromic*) width of F_n is finite.

The goal of the present paper is to determine the primitive and palindromic widths of a free group. The main results of the paper show that both widths of a finitely generated non-abelian free group are infinite, and for infinitely generated free groups, the palindromic width is infinite, too, whereas the primitive width is not (one easily checks that the primitive width of an infinitely generated free group is two). Note that, although our results do not solve the problem of definability of the subgroup of inner

V. Bardakov was partially supported by the RFFI grant # 02-01-01118.

V. Tolstykh was supported by a NATO PC-B grant via The Scientific and Technical Research Council of Turkey (TÜBITAK).

automorphisms in the automorphism group of a finitely generated free non-abelian group, they can be considered an evidence of its difficulty.

In Section 1 we examine the palindromic width of a free group. Recall that given a basis X of a free group F, a reduced word $w \in F$ is called a *palindrome* if w reads the same left-to-right and right-to-left as a word in letters from $X^{\pm 1}$. The *palindromic width* of F is therefore the width of F with respect to the palindromes associated with a given basis of F. Palindromes of free groups have already proved useful in studying various aspects of combinatorial group theory: for instance, in [5, 9], palindromic automorphisms of free groups are studied and in [11], palindromes are used in a description of automorphisms of a two-generator free group.

Using standard methods developed earlier for the study of verbal subgroups of free contructions, we prove that the palindromic width of any free group is infinite (Theorem 1.1). Then we discuss relations between the primitive and palindromic widths of a two-generator free group F_2. In this special case we establish the following result which seems to be of independent interest: any primitive element of F_2 is a product of at most two palindromes. (Here we mention, in passing, an interesting related fact [16]: any primitive element of a free associative algebra of rank 2 is palindromic.) This, together with Theorem 1.1, implies that the primitive width of F_2 is infinite.

In Section 2 we generalize the latter result by proving that the primitive width of a finitely generated free group $F_n, n \geqslant 2$, is infinite (Theorem 2.1). Actually, a stronger result is established: the width of F_n relative to the set W_n of elements of F_n whose Whitehead graph has a cut vertex is infinite. Recall that, according to a classical result of Whitehead, the set W_n contains all primitive elements of F_n [21].

To conclude the Introduction, we mention several interesting, in our opinion, problems motivated by the results of this paper.

Problem 1. Let F_n be a free group of finite rank $n \geqslant 2$. Is there an algorithm to determine the primitive length of an element w of F_n?

Problem 2. The same question for the palindromic length of an element of F_n.

There are two interesting related problems that can be formulated in a different language: is the Dehn function relative to the set of primitive (resp. palindromic) elements recursive? Here, by somewhat abusing the language, by the Dehn function relative to a generating set M of a group G with a fixed generating set S we mean a function $f_{S,M}(n)$ equal to the maximum length $l_M(w)$ over all $w \in G$ such that $l_S(w) = n$. With respect to Problems 1 and 2, we naturally assume that S is an arbitrary but fixed basis of F_n. If these Dehn functions turn out to be recursive, it would be interesting to find out what they are. It is clear that positive answers to Problems 1 and 2 would imply that the corresponding Dehn functions are recursive.

We mention here that Grigorchuk and Kurchanov [10] have reported an algorithm to determine the length of an element w of F_n with respect to the set of all conjugates of elements of a *fixed basis* of F_n.

Finally, we note that the primitive width can be defined not only for free groups, but for all relatively free groups as well. In particular, it is easy to see that the primitive width of any free abelian group and any free nilponent group is two. Smirnova [17] proved that the primitive width of any free metabelian group of rank $\geqslant 2$ is at most four. Lapshina [12] generalized this to relatively free groups with nilpotent commutator subgroup. Later she also proved that the primitive width of the free solvable group of rank n and class s is at most $4 + (s-1)(n-1)$. The same is true, in fact, for any free polynilpotent group of rank n and class $(k_1, ..., k_s)$ [13]. All this motivates the following

Problem 3. Describe relatively free groups whose primitive width is infinite.

The authors are grateful to Oleg Belegradek and Oleg Bogopolsky for helpful discussions.

1. The palindromic width of a free group

Let F be a free group with a basis X. We call a word $w \in F$ a *palindrome in letters* X if

$$w = x_{i_1} x i_2 \dots x i_k = x i_k \dots x i_2 x_{i_1},$$

where $x_{i_j} \in X^{\pm 1}$ and $x_{i_j} x_{i_{j+1}} \neq 1$ for all $j = 1, 2, \dots, k-1$. Equivalently, it can be said that $w \in F$ is a palindrome if and only if w coincides with its *reverse* word [5]. For example, if x_1, x_2, x_3 are distinct elements of X, then the word

$$x_1^{-2} x_2^3 x_3^{-4} x_2^3 x_1^{-2}$$

is a palindrome.

Since elements of a basis X of a free group F are palindromes in letters X, the set of all palindromes generates F, and we can define the *palindromic width* of F as the width relative to the set of palindromes.

Theorem 1.1. *The palindromic width of a non-abelian free group F is infinite.*

Proof. A standard idea of showing that the width of a given group relative to a given generating set is infinite is constructing a *quasi-homomorphism* $\Delta : F \to \mathbf{Z}$ satisfying the inequality

$$\Delta(uw) \leqslant \Delta(u) + \Delta(w) + \text{const} \tag{1.1}$$

for all $u, w \in F$. After constructing such a quasi-homomorphism, we prove, for all $k \in \mathbf{N}$, that there is a bound c_k such that

$$\Delta(w) \leqslant c_k$$

for all $w \in F$ that are products of at most k palindromes. This will surely hold if the values of Δ with (1.1) are reasonably bounded on palindromes (say, $\Delta(p) = 0$ for all palindromes p); one can say then, somewhat informally, that Δ *recognizes* palindromes. Finally, we shall find a sequence (w_n) of elements of F with

$$\lim_{n\to\infty} \Delta(w_n) = +\infty.$$

Clearly, having all these requirements on Δ satisfied, one obtains that the palindromic width of F is infinite, since for all n with $\Delta(w_n) > c_k$, the word w_n is not a product of k palindromes.

We shall now define a required quasi-homomorphism by induction on the number of syllables of a reduced word $w \in F$ in letters X.

1) Suppose w has exactly one syllable, then

$$\Delta(w) = 0.$$

2) Let now

$$w = v_1 \dots v_{m-1} v_m,$$

where $m > 1$ and v_k are syllables of w. Assume that

$$v_{m-1} = a^{\pm k}, v_m = b^{\pm l},$$

where $a, b \in X$, $a \neq b$, and $k, l \in \mathbf{N}$. Then

$$\Delta(w) = \Delta(v_1 \dots v_{m-1}) + \mathrm{sign}(l - k).$$

The function sign : $\mathbf{Z} \to \{-1, 0, 1\}$ takes the value 1 at positive integers, the value -1 at negative integers and $\mathrm{sign}(0) = 0$.

For example, if x_1, x_2, x_3 are distinct elements of X, then

$$\begin{aligned}\Delta(x_1^2 x_2^{-3} x_3^4 x_2^{-3} x_1^2) &= \mathrm{sign}(2-3) + \mathrm{sign}(3-4) + \mathrm{sign}(4-3) + \mathrm{sign}(3-2) \\ &= -1 - 1 + 1 + 1 = 0.\end{aligned}$$

One readily verifies the following properties of Δ :

Claim 1.2. (i) *For any word w in F,*

$$\Delta(w) + \Delta(w^{-1}) = 0;$$

(ii) *for every palindrome p in letters X,*

$$\Delta(p) = 0.$$

Note that not only palindromes give the value of Δ equal to 0. For instance,

$$\Delta(x_1 x_2 \dots x_n) = 0,$$

where x_i are distinct elements of X.

Now we are going to show that Δ is a quasi-homomorphism.

Lemma 1.3. *For any $u, w \in F$*

$$\Delta(uw) \leqslant \Delta(u) + \Delta(w) + 6.$$

Proof. Suppose that

$$\begin{aligned}u &= u_0 t^k z^{-1}, \\ w &= z t^m w_0,\end{aligned}$$

where

- u_0, w_0 are reduced;
- $t \in X$ and $k + m \neq 0$;
- neither u_0 ends with t, nor w_0 begins with t.

Then we have

$$\Delta(u) = \Delta(u_0) + \varepsilon_1 + \varepsilon_2 + \Delta(z^{-1}),$$
$$\Delta(w) = \Delta(z) + \varepsilon_3 + \varepsilon_4 + \Delta(w_0),$$

where $\varepsilon_1, \ldots, \varepsilon_4 \in \{-1, 0, 1\}$. Furthermore,

$$\Delta(uw) = \Delta(u_0 t^{k+m} w_0) = \Delta(u_0) + \varepsilon_5 + \varepsilon_6 + \Delta(w_0),$$

and again $\varepsilon_5, \varepsilon_6 \in \{-1, 0, 1\}$. Clearly,

$$\varepsilon_5 + \varepsilon_6 \leqslant \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + 6.$$

Hence by Claim 1.2

$$\Delta(uw) \leqslant \Delta(u) + \Delta(w) + 6,$$

as needed. □

If now w is a product of at most k palindromes, then a straightforward induction argument using Lemma 1.3 proves that

$$\Delta(w) \leqslant 6k - 6.$$

We are going to find a sequence (w_n) of words of F such that

$$\Delta(w_n) = n - 1$$

for all $n \geqslant 1$. This will imply that for any $n > 6k - 5$ the word w_n is not a product of k palindromes, and the palindromic width of F is therefore infinite. A particular choice of w_n like that is as follows :

$$w_n = x_1 x_2 x_1^2 x_2^2 \ldots x_1^n x_2^n,$$

where $n \geqslant 1$ and x_1, x_2 are fixed distinct elements of X.

This completes the proof of Theorem 1.1. □

In the remainder of this section we discuss the relation between the palindromic and primitive widths of two-generator free groups. We start with an elementary observation from [9] which provides a useful description of palindromes.

Lemma 1.4. *Let F be a free group with a basis X. Suppose that θ is an involution of* $\mathrm{Aut}(F)$ *that inverts each element $x \in X$. Then $w \in F$ is a palindrome in letters X if and only if θ inverts w.*

Remark 1.5. Note that if p is a palindrome in letters X and u is any word of $F(X)$, then by Lemma 1.4 the words $\theta(u)u^{-1}$ and $\theta(u)pu^{-1}$ are also palindromes in letters X (cf. [19, Lemma 2.3].)

Lemma 1.6. *Any primitive element of a two-generator free group F_2 is a product of at most two palindromes.*

Proof. Let $\{x, y\}$ be a basis of F_2. Let τ_w denote the inner automorphism determined by $w \in F_2$. Suppose that θ is the automorphism of F_2 that inverts both x and y. Since θ induces the automorphism id_A on the abelianization $A = F_2/[F_2, F_2]$ of F_2, any product of the form

$$\theta\sigma\theta\sigma^{-1},$$

where σ is an arbitrary automorphism of F_2, induces the trivial automorphism of A and hence, by a well known result of Nielsen (see e.g. [14, I.4.13]), it is an inner automorphism determined by some $p \in F_2$:

$$\theta\sigma\theta\sigma^{-1} = \tau_p.$$

We claim that p is a palindrome. Indeed,

$$\begin{aligned}\theta\tau_p\theta &= \tau_{\theta(p)} = \theta(\theta\sigma\theta\sigma^{-1})\theta \\ &= \sigma\theta\sigma^{-1}\theta = \tau_p^{-1} = \tau_{p^{-1}}.\end{aligned}$$

Then $\theta(p) = p^{-1}$ and by Lemma 1.4 p is a palindrome; we denote p by $p(\sigma)$.

Let U be the Nielsen automorphism that acts on $\{x, y\}$ as follows:

$$\begin{aligned}U(x) &= x, \\ U(y) &= xy.\end{aligned}$$

One easily checks that

$$\theta U\theta U^{-1} = \tau_x.$$

Then we have

$$\begin{aligned}\tau_{p(\sigma U)} &= \theta\sigma U\theta U^{-1}\sigma^{-1} = \theta\sigma\theta(\theta U\theta U^{-1})\sigma^{-1} \\ &= \theta\sigma\theta\tau_x\sigma^{-1} = \theta\sigma\theta\sigma^{-1}\sigma\tau_x\sigma^{-1} \\ &= \tau_{p(\sigma)}\tau_{\sigma(x)}.\end{aligned}$$

This tells us that

$$\sigma(x) = p(\sigma)^{-1}p(\sigma U),$$

that is, any element in the orbit of x under $\mathrm{Aut}(F_2)$ (in other words, any primitive element of F_2) is a product of two palindromes in letters x, y. □

As a corollary we immediately obtain the following fact which will be generalized in the next section. By the *primitive width* of a free group F we mean the width of F relative to the set of all primitive elements.

Corollary 1.7. *The primitive width of a two-generator free group F_2 is infinite.*

Proof. The result follows immediately from Theorem 1.1 and Lemma 1.6. □

Remark 1.8. (i) It is also possible to derive a fact similar to Lemma 1.6 from one of the results in the paper [11] by Helling. Indeed, let a be a primitive element of $F_2 = F(x, y)$. It then follows from the Theorem on p. 613 of [11] that there are a palindrome p in letters x, y and an element $z \in F_2$ such that

$$a = zy^{-1}pxz^{-1}$$

or

$$a = zx^{-1}pyz^{-1}.$$

On the other hand, for instance, in the first case we have

$$a = zy^{-1}\theta(z^{-1}) \cdot \theta(z)pz^{-1} \cdot zx\theta(z^{-1}) \cdot \theta(z)z^{-1},$$

where θ is the automorphism of F_2 that inverts both x and y. Therefore, by Remark 1.5, an arbitrary primitive element of F_2 is a product of at most four palindromes.

(ii) There is an important consequence of the cited result by Helling which seems to pass unnoticed and which improves the description of primitive elements of F_2 given in [6]. It is proved there that the conjugacy class of every primitive element of $F_2(x, y)$ contains either an element of the form

$$a = xy^{m_1}xy^{m_2}\dots xy^{m_k}, \tag{1.2}$$

where $m_k \in \{n, n+1\}$ for a suitable fixed natural n, or an element obtained from a by (a) permutation of the basis elements x, y, or (b) by inversion of one or both basis elements. It is then clear in view of the result by Helling that the distribution of exponents $n, n+1$ in (1.2) must obey a rather strict rule: there must be a cyclic permutation a' of the word a such that $xa'y^{-1}$ or $x^{-1}a'y$ is a palindrome.

(iii) In sharp contrast with the two-generator case, the palindromic length of primitive elements in a free group F_n of rank $n > 2$ *cannot* be uniformly bounded. Indeed, if there is a word $w_k(x, y)$ in letters x, y which is not a product of at most k palindromes in $F_2 = F(x, y)$, then, for instance, the word $zw_k(x, y)$, a primitive element of $F_3 = F(x, y, z)$, cannot be written as a product of at most k palindromes. It follows immediately from the fact that the homomorphism

$$x \to x, y \to y, z \to 1$$

from $F(x, y, z)$ to $F(x, y)$ takes palindromes to palindromes.

To conclude this section, we discuss a possible generalization of our results here.

The notion of a palindromic word, or a palindrome, can be defined for free products of groups as follows. Given a free product

$$G = \prod_{i\in I}^{*} G_i, \tag{$*$}$$

a reduced word g of G is called a *palindrome* associated with the decomposition $(*)$ if, when written in syllables from G_i, g "reads the same left-to-right and right-to-left". The palindromic width of G is the width of G relative to the set of palindromes.

It seems that any free product of groups which is not a product of two cyclic groups of order two has infinite palindromic width. We show this here in the case of a free product of two *infinite* groups. Using the "homomorphism argument", as in Remark 1.8 (iii), one can see that any free product of groups at least two of which are infinite has infinite palindromic width. The proof is, in fact, a modification of the proof of Theorem 1.1.

Suppose that $G = A * B$ and both groups A, B are infinite. Then we consider a surjective map $d : A \cup B \to \mathbf{N}$ such that $d(A) = d(B) = \mathbf{N}$. We define a quasi-homomorphism Δ recognizing palindromes as follows:

1) $\Delta(v) = 0$, if $v \in A, B$;

2) if $g = v_1 \dots v_n$ is a reduced word written in syllables and $n > 1$, then

$$\Delta(g) = \Delta(v_1 \dots v_{n-1}) + \mathrm{sign}(d(v_n) - d(v_{n-1})).$$

One then verifies that Δ is indeed a quasi-homomorphism. Furthermore, let (a_n) and (b_n) be sequences of elements of A and B respectively with $d(a_n) = d(b_n) = n$ for all

$n \in \mathbf{N}$. Then

$$\Delta(a_1 b_1 a_2 b_2 \dots a_n b_n) = n - 1.$$

The reader will easily supply missing details in our sketch.

2. The primitive width of a free group

Theorem 2.1. *Let F be a free group. Then the primitive width of F is infinite if and only if F is finitely generated. The primitive width of any infinitely generated free group is two.*

Remark 2.2. In contrast, the palindromic width of an infinitely generated free group is infinite, as we have seen above. Thus the set of palindromes in any infinitely generated free group appeares to be quite "sparse" compared to the set of primitive elements.

Proof. We begin with the proof of our second, almost obvious, statement. Suppose that X is an infinite basis of F. Take an arbitrary w in F and assume that w is a word in letters $y_1, \dots, y_k$ of X: $w = w(y_1, \dots, y_k)$. Since X is infinite, there is a basis letter x which is different from all $y_1, \dots, y_k$. We have then

$$w = x^{-1} \cdot xw(y_1, \dots, y_k).$$

Clearly, both words x^{-1} and $xw(y_1, \dots, y_k)$ are primitive elements of F.

Let now $F = F_n$ be a finitely generated free group with a free basis $X = \{x_1, \dots, x_n\}$. Let w be a (reduced) word from F. The *Whitehead graph* WG_w of w is constructed as follows: the vertices of WG_w are the elements of the set $X^{\pm 1}$, and the edges are determined by the pairs (a, b^{-1}), where $a, b \in X^{\pm 1}$ are such that there is an occurence of the subword ab in w. Note that, like in most recent papers that use the Whitehead graph (see e.g. [3]), we are in fact using a simplified version of the graph introduced by Whitehead in [21]. The principal result about the Whitehead graph of a primitive element of a free group is the following:

Theorem 2.3 (Whitehead, [21])**.** *The Whitehead graph of a primitive element of a free group has a cut vertex.*

Recall that a vertex v of a graph Γ is said to be a *cut vertex* if removing the vertex v along with all its adjacent edges from Γ increases the number of connected components of the graph. The fact that the above theorem is also true for the simplified Whitehead graph is an easy corollary of Theorem 2.4 from the paper [18] by Stallings.

Now we give a proof of the existence of words of F that cannot occur as proper subwords of primitive elements of F. We claim that any word $u \in F$ whose Whitehead graph is Hamiltonian, that is, such that all the vertices of WG_u can be included into a simple circuit, meets our condition. Indeed, if u occurs in a word w as a proper subword, then the graph WG_u is a subgraph of WG_w, and hence WG_w contains no cut vertices since there are no cut vertices in WG_u (because the latter graph is Hamiltonian). Thus, by the Whitehead theorem, w is not primitive.

An example of such a word u can be easily found: for instance, the Whitehead graph of the word

$$\boldsymbol{u} = (x_1^2 \dots x_n^2)^2$$

is Hamiltonian.

Lemma 2.4. *Let k be a natural number, and $\boldsymbol{u} = (x_1^2 \dots x_n^2)^2$. The element $\boldsymbol{u}^{2k}$ is not a product of at most k primitive elements of F.*

Proof. Let $\mathrm{sl}(w)$ denote the syllabic length of a word $w \in F$ with respect to the basis X of F. Suppose that

$$\boldsymbol{u}^{2k} = p_1 \dots p_m,$$

where $m \leqslant k$ and $p_1, \dots, p_m$ are primitive elements of F. After possible reductions the product $p_1 \dots p_m$ turns into the product $p_1' \dots p_n'$, where p_i' is a subword of p_i, and the latter product has the following property:

(2.1) for every $i = 1, \dots, m-1$ the last syllable of p_i' and the first syllable of p_{i+1}' do not annihilate each other; in other words,

$$\mathrm{sl}(p_i' p_{i+1}') = \mathrm{sl}(p_i') + \mathrm{sl}(p_{i+1}') + \varepsilon_i,$$

where ε_i equals 0 or -1.

We then have that

$$\mathrm{sl}(\boldsymbol{u}^{2k}) = \mathrm{sl}(p_1' \dots p_m'),$$

or

$$4nk = \mathrm{sl}(p_1') + \varepsilon_1 + \dots + \varepsilon_{m-1} + \mathrm{sl}(p_m'),$$

where $\varepsilon_i = 0, -1$. Letting l denote the sum $-\varepsilon_1 - \dots - \varepsilon_{m-1}$, we get

$$4nk + l = \mathrm{sl}(p_1') + \dots + \mathrm{sl}(p_m').$$

Then there is an index j such that

$$\mathrm{sl}(p_j') \geqslant \frac{4nk + l}{m} \geqslant \frac{4nk + l}{k} \geqslant 4n \geqslant 2n + 2.$$

The condition (2.1) implies therefore that p_j' contains at least $2n$ consecutive syllables of $\boldsymbol{u}$. However, for the subword z of $\boldsymbol{u}$ determined by these syllables the Whitehead graph is clearly Hamiltonian, because z is the result of a cyclic permutation of the syllables of $\boldsymbol{u}$. It follows that p_j has z as a proper subword, and then it is not primitive. This contradiction completes the proof of the lemma. □

According to Lemma 2.4, there are elements of F of arbitrarily large primitive length. Therefore the primitive length of F is infinite. □

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