Test polynomials, retracts, and the Jacobian conjecture

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Abstract. Let $K[x, y]$ be the algebra of two-variable polynomials over a field K. A polynomial $p = p(x, y)$ is called a *test polynomial* for automorphisms if, whenever $\varphi(p) = p$ for a mapping φ of $K[x, y]$, this φ must be an automorphism. Here we show that $p \in \mathbb{C}[x, y]$ is a test polynomial if and only if p does not belong to any proper retract of $\mathbb{C}[x, y]$. This has the following corollary that may have application to the Jacobian conjecture: if a mapping φ of $\mathbb{C}[x, y]$ with invertible Jacobian matrix is "invertible on one particular polynomial", then it is an automorphism. More formally: if there is a non-constant polynomial p and an injective mapping ψ of $\mathbb{C}[x, y]$ such that $\psi(\varphi(p)) = p$, then φ is an automorphism. We also show that if φ is a counterexample to the Jacobian conjecture for $\mathbb{C}[x, y]$, then $\varphi(x)$ must be a test polynomial.

1. Introduction

Let $K[x, y]$ be the algebra of two-variable polynomials over a field K of characteristic 0. A subalgebra R of $K[x, y]$ is called a *retract* if there is an idempotent homomorphism π of $K[x, y]$ (called a *retraction* or a *projection*) such that $\pi(K[x, y]) = R$.

There are several equivalent descriptions of retracts of $K[x, y]$ known by now:

(i) $K[x, y] = R \oplus I$ for some ideal I of $K[x, y]$.

(ii) $K[x, y]$ is a projective extension of R in the category of K-algebras.

(iii) By a theorem of Costa [**2**], every proper retract of $K[x, y]$ (i.e., one different from $K[x, y]$ and K) is of the form $K[p]$ for some $p = p(x, y) \in K[x, y]$. The authors earlier proved [**9**] that there exists an automorphism of $K[x, y]$ which takes $p(x, y)$ to $x + y \cdot q(x, y)$ for some $q(x, y) \in K[x, y]$,

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and every polynomial of the form $x + y \cdot q(x, y)$ generates a proper retract of $K[x, y]$.

(iv) (see [**9**]) $p(x, y)$ generates a retract of $K[x, y]$ if and only if there is an endomorphism of $K[x, y]$ which takes $p(x, y)$ to x.

(v) (see [**3**]) $p(x, y)$ belongs to a proper retract of $\mathbb{C}[x, y]$ if and only if $p(x, y)$ is fixed by some endomorphism of $\mathbb{C}[x, y]$ with nontrivial kernel.

Recently retracts have found another application in a general setup of arbitrary free algebras and groups in relation with test elements, introduced in [**8**]. In general, an element g of a group or an algebra F is a test element if any endomorphism of F fixing g is actually an automorphism. It is easy to see that a test element does not belong to any proper retract of F; a remarkable result of Turner [**10**] says that, if F is a free group, then the converse is also true. Thus, an element of a free group F is a test element if and only if it does not belong to any proper retract of F.

Here we establish a similar characterization of test polynomials in $\mathbb{C}[x, y]$:

Theorem 1. A polynomial $p \in \mathbb{C}[x, y]$ is a test polynomial if and only if p does not belong to any proper retract of $\mathbb{C}[x, y]$.

Our proof uses several recent results, in particular, a result of Drensky and Yu [**3**] mentioned in the item (v) above. Crucial for our proof is the following result of independent interest.

Theorem 2. Let φ be an injective endomorphism of $\mathbb{C}[x, y]$ which is not an automorphism. Suppose that $\varphi(p) = p$ for some non-constant polynomial $p \in \mathbb{C}[x, y]$. Then $p \in \mathbb{C}[q]$, where q is a coordinate polynomial of $\mathbb{C}[x, y]$. In particular, p belongs to a proper retract of $\mathbb{C}[x, y]$.

Recall that $q = q(x, y)$ is a *coordinate polynomial* of $\mathbb{C}[x, y]$ if it can be taken to x by an automorphism of $\mathbb{C}[x, y]$.

We also use results of Shestakov and Umirbaev [**7**] on estimating degrees of polynomials in two-generated subalgebras of $K[x, y]$. Another ingredient is a result of Kraft [**5**] concerning the subalgebra $\varphi^\infty(\mathbb{C}[x, y]) = \cap_{k=1}^{\infty} \varphi^k(\mathbb{C}[x, y])$.

Theorems 1, 2 have the following corollary:

Corollary. Let φ be an endomorphism of $\mathbb{C}[x, y]$ with invertible Jacobian matrix. If there is a non-constant polynomial $p \in \mathbb{C}[x, y]$ and an injective mapping ψ such that $\psi(\varphi(p)) = p$, then φ is an automorphism of $\mathbb{C}[x, y]$.

This strengthens our earlier result [**9**, Corollary 1.7], where we showed that, if φ has invertible Jacobian matrix, then $\varphi(p) = p$ implies that φ is an automorphism of $\mathbb{C}[x, y]$.

To conclude the Introduction, we raise a problem motivated by results of this paper:

Problem. Suppose $p \in \mathbb{C}[x, y]$ is a test polynomial and φ is an injective mapping of $\mathbb{C}[x, y]$. Is $\varphi(p)$ necessarily a test polynomial?

It is interesting to note that, by a result of Jelonek [**4**], a "generic" polynomial of degree ≥ 4 is a test polynomial.

2. Proof of Theorem 2

We consider the following two principal cases.

Case I. There is a coordinate polynomial in $\varphi(\mathbb{C}[x, y])$.

Case II. There are no coordinate polynomials in $\varphi(\mathbb{C}[x, y])$.

In Case I, consider two subcases:

(1) φ is *not birational*, i.e., does not induce an automorphism of the field of fractions.

Then, by a result of Kraft [**5**, Lemma 1.3], $\varphi^{\infty}(\mathbb{C}[x, y])$ is either $\mathbb{C}$ or $\mathbb{C}[f]$, where $f = f(x, y)$ is some polynomial. Obviously, if $\varphi(p) = p$, then $p \in \varphi^{\infty}(\mathbb{C}[x, y])$. We are therefore going to focus on the case $\varphi^{\infty}(\mathbb{C}[x, y]) = \mathbb{C}[f]$ and show that, if φ is injective, then $p \in \mathbb{C}[q]$, where q is a coordinate polynomial.

Now suppose $r = r(x, y)$ is a coordinate polynomial in $\varphi(\mathbb{C}[x, y])$, and let $r = \varphi(s(x, y))$. Then, since φ is injective, the polynomial $s = s(x, y)$ must be coordinate, too, by the result of [**1**]. Therefore, upon changing generating set of $\mathbb{C}[x, y]$ if necessary, we may assume that $r = \varphi(x)$. Furthermore, we can replace φ with its conjugate by an arbitrary automorphism, say α, i.e., with $\psi = \alpha\varphi\alpha^{-1}$, and at the same time replace p with $p_1 = \alpha(p)$. Then we have:

$$\psi(p_1) = \alpha\varphi\alpha^{-1}(\alpha(p)) = \alpha(p) = p_1.$$

Therefore, the pair (ψ, p_1) has the same properties that the pair (φ, p) does, namely, ψ is injective but not birational, and $\psi(p_1) = p_1$; in particular, $p_1 \in \psi^{\infty}(\mathbb{C}[x, y])$. By choosing α appropriately, we can also have $\psi(x) = x$, thus getting $x \in \psi^{\infty}(\mathbb{C}[x, y])$. Then, if p (and therefore p_1) does not belong to $\mathbb{C}[q]$ for any coordinate polynomial q, $\psi^{\infty}(\mathbb{C}[x, y])$ cannot be of the form $\mathbb{C}[f]$, which is in contradiction with the result of Kraft mentioned above. This completes case (1).

(2) φ is birational, i.e., induces an automorphism of the field of fractions. Again, as in the case (1) above, we deduce from [**1**] that φ must take some coordinate polynomial to coordinate. Thus, upon changing generating set of $\mathbb{C}[x, y]$ if necessary, we may assume that φ takes x to u, and y to $v \cdot f(u)$, where $\mathbb{C}[u, v] = \mathbb{C}[x, y]$, and $f(u)$ is a non-constant polynomial (otherwise, φ would be an automorphism).

Now let

$$p = p(x, y) = \sum_{i,j} c_{ij} x^i y^j.$$

Then

$$\varphi(p) = \sum_{i,j} c_{ij} u^i v^j (f(u))^j.$$

Let $x^r y^s$ be the highest term of $p(x, y)$ in the "pure lex" order with $y > x$. Then in $\varphi(p)$, the highest term is that of $\varphi(x^r y^s)$ because the y-degree of $\varphi(y)$ is not lower than that of $\varphi(x)$. Furthermore, the highest term of $\varphi(x^r y^s)$ must have the y-degree at least s since otherwise, one would have both u and v of y-degree equal to 0, which is impossible.

If the y-degree of $\varphi(x^r y^s)$ is $> s$, this gives a contradiction with $\varphi(p) = p$. Now suppose the y-degree of $\varphi(x^r y^s)$ is *exactly* s. This is only possible if the y-degree of v is 1 and the y-degree of u is 0. Then, arguing as in the case (1) above, we may assume that $\varphi(x) = x$. Therefore, $\varphi(y) = (y + g(x)) \cdot f(x)$. Then from $\varphi(p) = p$ we get:

$$\sum_{i,j} c_{ij} x^i y^j = \sum_{i,j} c_{ij} x^i (y + g(x))^j (f(x))^j.$$

Again we use the "pure lex" order with $y > x$ to focus on the monomial of highest degree on either side, but this time we compare the x-degrees of these highest-degree monomials. We see that these x-degrees cannot be equal unless $f(x)$ is a constant, contradicting the assumption. This completes the proof in Case I.

In Case II, we are going to prove the following somewhat stronger statement:

Proposition. Let φ be an injective endomorphism of $\mathbb{C}[x, y]$, and suppose that there are no coordinate polynomials in $\varphi(\mathbb{C}[x, y])$. Then $\varphi^\infty(\mathbb{C}[x, y]) = \mathbb{C}$.

Proof. Let $\varphi(x) = u = u(x, y)$, $\varphi(y) = v = v(x, y)$, and let $D(u, v)$ denote the determinant of the Jacobian matrix of φ. Since φ is injective, $D(u, v) \neq 0$. Now there are two cases:

(1) $\deg(D(u, v)) = 0$, i.e., $D(u, v)$ is a non-zero constant. Then, by a result of Kraft [**5**], we have $\varphi^\infty(\mathbb{C}[x, y]) = \mathbb{C}$.

(2) $\deg(D(u, v)) > 0$. Note that for any $k \geq 1$, there are no coordinate polynomials in $\varphi^k(\mathbb{C}[x, y])$. Indeed, if there were a coordinate polynomial in $\varphi^k(\mathbb{C}[x, y])$, then, by the result of [**1**], there would have to be a coordinate polynomial in $\varphi^{k-1}(\mathbb{C}[x, y])$. This would lead to a contradiction with the assumption that there are no coordinate polynomials in $\varphi(\mathbb{C}[x, y])$.

Let $\varphi^k(x) = u^{(k)}$, $\varphi^k(y) = v^{(k)}$. Then from $\deg(D(u, v)) > 0$ and from the "chain rule" we get $\deg(D(u^{(k)}, v^{(k)})) \geq k$. Now the Proposition will follow from the lemma below. Before we get to it, we need one more definition.

We call a pair (p, q) of polynomials from $K[x, y]$ *elementary reduced* if the sum of their degrees cannot be reduced by a (non-degenerate) linear transformation or a transformation of one of the following two types:

(i) $(p, q) \longrightarrow (p + \mu \cdot q^k, q)$ for some $\mu \in K^*$; $k \geq 2$;

(ii) $(p, q) \longrightarrow (p, q + \mu \cdot p^k)$.

Now we are ready for our

Lemma. Let $p = p(x,y)$ and $q = q(x,y)$ be two algebraically independent polynomials such that the pair (p,q) is elementary reduced. Let $n = \deg(p) < m = \deg(q)$; $m,n \geq 2$, $\deg(D(p,q)) \geq k$. Let $w = w(x,y) \in \mathbb{C}[p,q]$). Then, unless w is a linear combination of p and q, one has $\deg(w) > min(n,k)$.

Proof. The proof here is based on a result of Shestakov and Umirbaev [**7**, Theorem 3]. Let $N = N(p,q) = \frac{mn}{g.c.d.(n,m)} - m - n + \deg(D(p,q)) + 2$. Following [**7**], we may assume that the highest homogeneous parts of p and q are algebraically dependent; otherwise, $\deg(w) > n$ is immediate (unless w is a linear combination of p and q). Then $\frac{mn}{g.c.d.(n,m)} - m - n \geq 0$. Indeed, if $g.c.d.(n,m) = n$, then the pair (p,q) would not be elementary reduced, contradicting the assumption. If $g.c.d.(n,m) < n$, then $\frac{n}{g.c.d.(n,m)} \geq 2$, therefore $\frac{mn}{g.c.d.(n,m)} \geq 2m$, hence $\frac{mn}{g.c.d.(n,m)} - m - n \geq 0$.

Thus, from now on we assume $N = N(p,q) \geq \deg(D(p,q)) + 2$.

Suppose now that the y-degree of $w = w(x,y)$ is of the form $\frac{n}{g.c.d.(n,m)} \cdot b + r \neq 0$, where $0 \leq r < \frac{n}{g.c.d.(n,m)}$. Then, by [**7**, Theorem 3], we have

$$\deg(w(p,q)) \geq b \cdot N + mr.$$

If $b \neq 0$, this implies $\deg(w(p,q)) \geq N \geq k+2 > k$. If $b = 0$, then $r \neq 0$, implying $\deg(w(p,q)) \geq m > n$.

It remains to consider the case where the y-degree of $w = w(x,y)$ is 0. Then the x-degree of w must be nonzero; suppose it is of the form $\frac{m}{g.c.d.(n,m)} \cdot b_1 + r_1 \neq 0$, where $0 \leq r_1 < \frac{m}{g.c.d.(n,m)}$. Then, again by [**7**, Theorem 3], we have

$$\deg(w(p,q)) \geq b_1 \cdot N + nr_1.$$

As before, $b_1 \neq 0$ implies $\deg(w(p,q)) > k$. If $b_1 = 0$, then $r \geq 2$ because we assume that $w(x,y)$ is not linear. Then we have $\deg(w(p,q)) \geq 2n > n$, which completes the proof of the lemma. □

Continuing with the proof of the Proposition, we aim at showing that for any integer M, there is an integer k such that the degree of any polynomial in $\varphi^k(\mathbb{C}[x,y])$ is $> M$. The above lemma "almost" does it if we use it with $p = \varphi^k(x) = u^{(k)}$, $q = \varphi^k(y) = v^{(k)}$, but it has one extra condition on the pair (p,q) to be elementary reduced, whereas a pair $(u^{(k)}, v^{(k)})$ may not be elementary reduced. However, if we denote by $(\overline{u}^{(k)}, \overline{v}^{(k)})$ an elementary reduced pair obtained from $(u^{(k)}, v^{(k)})$ by elementary transformations, we shall have all conditions of the lemma satisfied for this pair while obviously $\mathbb{C}[\overline{u}^{(k)}, \overline{v}^{(k)}] = \mathbb{C}[u^{(k)}, v^{(k)}]$. In particular, the inequality $\deg(D(\overline{u}^{(k)}, \overline{v}^{(k)}))$ $\geq k$ follows from the fact that the mapping $x \to \overline{u}^{(k)}, y \to \overline{v}^{(k)}$ is a composition of φ^k with an automorphism α of $\mathbb{C}[x,y]$ in such a way that α is applied first. Therefore, the "chain rule" applied to this composition yields $\deg(D(\overline{u}^{(k)}, \overline{v}^{(k)}))$= $\deg(D(u^{(k)}, v^{(k)}))$. Thus, our lemma is applicable to the pair $(\overline{u}^{(k)}, \overline{v}^{(k)})$, which completes the proof of the Proposition and therefore of Theorem 2. □

3. Proof of Theorem 1 and Corollary

The "only if" part of Theorem 1 follows from a result of [**9**] rather easily. If $p = p(x, y)$ belongs to a proper retract $\mathbb{C}[q]$ of $\mathbb{C}[x, y]$, then, by [**9**], for some automorphism α, $\alpha(p)$ belongs to $\mathbb{C}[x + y \cdot u]$ for some polynomial $u = u(x, y)$. Then the mapping $x \to x + y \cdot u$, $y \to 0$ fixes the polynomial $x + y \cdot u$, and therefore also fixes $\alpha(p)$. Thus, $\alpha(p)$ is not a test polynomial, and neither is p.

For the "if" part of Theorem 1, suppose that p does not belong to any proper retract of $\mathbb{C}[x, y]$, and let $\varphi(p) = p$ for some mapping φ of $\mathbb{C}[x, y]$. Then, by the result of [**3**], φ must be injective. Then, by our Theorem 2, φ must be an automorphism, hence p is a test polynomial. □

Proof of Corollary. By way of contradiction, assume that φ is not an automorphism. Then, by our Theorem 2, $p \in \mathbb{C}[q]$, where q is a coordinate polynomial of $\mathbb{C}[x, y]$. Therefore, the composite mapping $\psi\varphi$ fixes a polynomial $f(q)$ in q. Then it is easy to see (by looking at the highest degree monomial in $f(q)$) that $\psi(\varphi(q)) = c \cdot q$ for some $c \in \mathbb{C}^*$, which implies, by the result of [**1**], that $\varphi(q)$ is coordinate. A mapping of $\mathbb{C}[x, y]$ with invertible Jacobian matrix that takes a coordinate polynomial to a coordinate polynomial is obviously an automorphism, a contradiction. □

In conclusion, we recall a result of [**9**, Theorem 1.3] saying that if, for a mapping φ of $\mathbb{C}[x, y]$ with invertible Jacobian matrix, $\varphi(x)$ generates a proper retract of $\mathbb{C}[x, y]$, then φ is an automorphism of $\mathbb{C}[x, y]$. Then, the case where $\varphi(x)$ belongs to a proper retract but does not generate it, can be ruled out since in that case, $\varphi(x) = f(p(x, y))$, where $p(x, y)$ generates a proper retract of $\mathbb{C}[x, y]$, and f is some one-variable polynomial of degree >1. The gradient of such a polynomial cannot form a row of any invertible Jacobian matrix, which can be easily seen from the "chain rule" applied to $f(p(x, y))$.

Therefore, by Theorem 1 of the present paper, if φ is a counterexample to the Jacobian conjecture for $\mathbb{C}[x, y]$, then $\varphi(x)$ must be a test polynomial. Perhaps a way to prove the Jacobian conjecture for $\mathbb{C}[x, y]$ could be through showing that the gradient of a test polynomial cannot form a row of any invertible Jacobian matrix. This is known to be the case with (non-commutative) partial derivatives of a test element of a free group of rank 2, see [**6**, Corollary 2.2.8].

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