Recognizing automorphisms of the free groups

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1 Introduction

Let F be the free group of a finite or countable infinite rank with a system $\{x_i\}_{i \in I}$ of free generators. When we restrict our attention to the free group of a finite rank $n \geqslant 2$, we usually write F_n for F.

Let ϕ be some endomorphism of F that takes x_i to y_i, $i \in I$. We will be concerned with deciding whether or not this ϕ is actually an automorphism of the group F. For the groups F_n, there are classical algorithms for deciding this due to Nielsen and Whitehead (see [8]). There is also a nice "inverse function theorem" of Birman [1]. But the application of these algorithms is limited to the case when the elements y_i are given explicitly. Hence it is desirable to have some more flexible approaches that would enable one, in particular, given y_i depending on some parameters, to decide on which values of these parameters we do have an automorphism of F_n — this problem arises, for example, when one considers the question of lifting an automorphism of a group given by presentation F/R, to an automorphism of F. Also, it is desirable to have a procedure that would be appropriate for some other groups (not necessarily free) as well.

The most well-known example of such an approach is due to Nielsen. In [10], he considered the free group F_2 and proved that y_1 and y_2 generate F_2 if and only if the commutator $[x_1, x_2]$ is conjugate to $[y_1, y_2]$ or $[y_2, y_1]$. The original proof of this fact is quite complicated; the proof of the "only if" part is based on the actual characterization of the group $\mathrm{Aut}\, F_2$ which is given in the same paper [10]; the geometric proof of the "if" part Nielsen attributes to Dehn. Later on, Magnus [9] has given a simpler proof of the "only if" part based on his famous Freiheitssatz.

The "commutator test" was recently shown to be valid for the free metabelian group of rank 2 [3], and also for a large class of groups of the form $F_2/[N', F_2]$ with N a normal subgroup of F_2 (the "if" part being valid with an arbitrary N) [6].

In the following Section 2, we give a very simple proof of the "if" part of Nielsen's test (actually in even a stronger form) as an illustration of our technique.

In Section 3, we are trying to generalize Nielsen's result as to be appropriate for a free group of arbitrary finite or countable infinite rank and prove the following

Theorem. *Let R be an arbitrary subgroup of a free group F. Then:*

(a) *generators* $x_1, \dots, x_n$ *of* F *belong to* R *if and only if the element* $[x_1, \dots, x_n]$ *belongs to* $\gamma_n(R)$*;*

(b) *for* $n = 2m$*, generators* $x_1, \dots, x_n$ *of* F *belong to* R *if and only if the element* $[x_1, x_2] \cdots [x_{2m-1}, x_{2m}]$ *belongs to* R^2*;*

(c) *for an arbitrary* $k \geqslant 2$*, generators* $x_1, \dots, x_n$ *of* F *belong to* R *if and only if the element* $x_1^k \cdots x_n^k$ *belongs to* $R'R^k$*.*

For any group G, by $\gamma_n(G)$ we denote the n-th term of its lower central series; we usually write G' for $\gamma_2(G)$. By $[x_1, \dots, x_n]$ we mean left-normed commutator: $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$, and $[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$ for $n \geqslant 3$. By G^k we denote the subgroup of G generated by the k-th powers of elements of G.

Now, given an endomorphism ϕ of the group F that takes x_i to y_i and considering the subgroup R of F generated by these y_i, this theorem yields several necessary and sufficient conditions for ϕ to be an automorphism of F. Denote the "test words" of parts (a), (b) and (c) by $u_1(n)$, $u_2(2m)$ and $u_3(n,k)$ respectively. The words $u_1(n)$ and $u_2(2m)$ are generalizations of the aforementioned Nielsen's "test word".

From the results of Zieschang [13], [14] it follows that if at least one of the words $u_2(2m)$ and $u_3(n,2)$ is a fixed point of an endomorphism ϕ of the group F_n, then this ϕ is an automorphism. These results were later generalized by Rosenberger [12]. Recently, Dold [2] has described in graph-theoretic terms a series of words with this property. This series includes, in particular, the words $u_2(2m)$ and $u_3(n,k)$ for an arbitrary $k \geqslant 2$.

These results naturally give rise to the problem of finding all the elements $u \in F_n$ with the property: $\phi(u) = u$ implies $\phi \in \operatorname{Aut} F_n$.

In order to include this problem in a general framework, we make the following

Definition. Let $u \in F$ be an arbitrary element. We define the *rank of* u to be the minimal number of free generators x_i on which the image of u under an automorphism of F can depend.

We note that it is algorithmically possible to determine the rank of an arbitrary $u \in F_n$ in view of [8, Propositions 4.25 and 5.4].

It is not difficult to see that in the group F_n, all the elements $u_1(n)$, $u_2(2m)$ and $u_3(n,k)$ (with $k \geqslant 2$) have rank n (in other words, none of them belongs to a proper free factor of F_n). We now denote by $A_G(u)$ the orbit of an element u of a group G under the action of the group $\operatorname{Aut} G$: $A_G(u) = \{\alpha(u),\ \alpha \in \operatorname{Aut}(G)\}$, and state the following problems. For notational convenience, we write here just F for F_n.

Problem 1. Find all elements u of the group F with the following property: an endomorphism ϕ of the group F is an automorphism if and only if $\phi(u) \in A_F(u)$. Or, equivalently: $\phi(u) = u$ implies $\phi \in \operatorname{Aut} F$.

We show that a necessary condition for such an element u is to have rank n and to belong to $F'F^k$ for some $k \geqslant 2$ (Proposition 3.2).

Another related problem of interest is:

Problem 2. Let ϕ be an endomorphism of the group F, and suppose for some element $u \in F$ one has $\phi(A_F(u)) \subseteq A_F(u)$. Is it true that ϕ is an automorphism of F?

An element u here might have an arbitrary rank; the most interesting particular case of this problem arises when u is a primitive element of F:

Problem 2a. Suppose ϕ takes every primitive element of the group F to a primitive one. Is it true that ϕ is an automorphism of F?

As usual, we call an element of a free group F *primitive* if it can be included in some free generating system of F. Note that for a free group F of infinite rank, the answer to the Problem 2a is obviously "no". For the group F_2, a simple but elegant proof of the "yes" has been given by S. Ivanov (verbal communication).

In Section 3, we also give an auxiliary result (Proposition 3.3) and an application of our theorem to solving equations in a free group (Corollary 3.4).

In the concluding Section 4, we show how one of the tests provided by our Theorem works for $F/[N', F]$ groups (we denote the elements of the free group and their natural images in a quotient group by the same letters without ambiguity):

Proposition 4.1. *Let N be a fully invariant subgroup of the free group $F = F_n$ of rank $n = 2m$. Let $G = F/[N', F]$, and suppose an endomorphism ϕ of G satisfies $\phi(u_2(2m)) \in A_G(u_2(2m))$. Then $\phi(G) = G$.*

This also gives one more way of proving the "if" part of Nielsen's "commutator test".

2 Preliminaries

We begin this section by introducing some more notation. Let $\mathbb{Z}F$ be the integral group ring of the group F and Δ_F its augmentation ideal, that is, the kernel of the natural homomorphism $\sigma\colon \mathbb{Z}F \to \mathbb{Z}$. More generally, when $R \subseteq F$ is a subgroup of F, we denote by Δ_R the left ideal of $\mathbb{Z}F$ generated by all elements of the form $(r-1)$, $r \in R$. In the case when R is a normal subgroup of F, Δ_R coincides with the two-sided ideal of $\mathbb{Z}F$ generated by the same elements.

In [4], Fox gave a detailed account of the differential calculus in a free group ring. We just give a brief summary here referring to the book [5] for more details.

The ideal Δ_F is a free right $\mathbb{Z}F$-module with a free basis $\{(x_i - 1)\}_{i \in I}$, and the right Fox derivations D_i are projections to the corresponding free cyclic direct summands. Thus any element $u \in \Delta_F$ can be uniquely written in the form $u = \sum_i (x_i - 1) D_i(u)$.

As the ideal Δ_F is a free left $\mathbb{Z}F$-module as well, one can define left Fox derivatives $d_i(u)$ accordingly, so that $u = \sum_i d_i(u)(x_i - 1)$.

One can extend these derivations to the whole $\mathbb{Z}F$ by linearity defining $D_i(1) = d_i(1) = 0$.

Furthermore, we have: $D_i(y^{-1}) = -D_i(y)y^{-1}$; $d_i(y^{-1}) = -y^{-1}d_i(y)$ for any $y \in F$, and if $y_1, y_2 \in F$, then $D_i([y_1, y_2]) = D_i(y_1)(y_2 - [y_1, y_2]) + D_i(y_2)(1 - y_2^{-1}y_1y_2)$; $d_i([y_1, y_2]) = y_1^{-1}y_2^{-1}(1 - y_2)d_i(y_1) + y_1^{-1}y_2^{-1}(y_1 - 1)d_i(y_2)$.

We need some technical lemmas to be used throughout Section 3. For proofs, see [5].

Lemma 2.1. *Let J be an arbitrary left (right) ideal of $\mathbb{Z}F$ and let $u \in \Delta_F$. Then $u \in \Delta_F J$ ($u \in J\Delta_F$) if and only if $D_i(u) \in J$ ($d_i(u) \in J$) for each $i \in I$.*

Lemma 2.2. *Let R be a subgroup of F, and let $y \in \gamma_{m+1}(R)$, $m \geqslant 1$. Then $y - 1 \in \Delta_F^m \Delta_R$.*

A simple proof of the next lemma can be found in [11].

Lemma 2.3. *Let G be an arbitrary group, K an arbitrary commutative ring with the unit; KG — the corresponding group ring. Let $v_1, \dots, v_m$ and u be elements of G. Suppose $(u - 1)$ belongs to the left ideal of KG generated by $(v_1), \dots, (v_m - 1)$. Then u belongs to the subgroup of G generated by $v_1, \dots, v_m$.*

We are now in position to give a simple proof of the "if" part of Nielsen's "commutator test" (in a stronger form):

Proposition 2.4. *Let y_1 and y_2 be some elements of the free group F_2, and let R be a subgroup of F_2 generated by y_1 and y_2. If the commutator $[x_1, x_2]$ belongs to R', then $R = F_2$.*

Proof. In view of Lemma 2.2, we can write:

$$[x_1, x_2] - 1 \in \Delta_F \Delta_R. \tag{1}$$

Now apply derivations D_1 and D_2 to both sides of (1):

$$x_2(1 - x_2^{-1}[x_1, x_2]) \in \Delta_R \tag{2}$$

$$1 - x_2^{-1}x_1x_2 \in \Delta_R. \tag{3}$$

By Lemma 2.3, (2) implies $x_2^{-1}[x_1, x_2] \in R$, and (3) implies $x_2^{-1}x_1x_2 \in R$, so we just have to show that these two elements generate F_2. Let $g = x_2^{-1}[x_1, x_2]$, $h = x_2^{-1}x_1x_2$, then $hg = x_2^{-1}h$, so $x_2 \in R$, and $x_1 = x_2hx_2^{-1}$, so $x_1 \in R$, and $R = F_2$. □

Remark 2.5. We can further strengthen Proposition 2.4 by claiming only $[x_1, x_2] \in R^2$ instead of $[x_1, x_2] \in R'$. Indeed, on replacing the integral group ring with the ring $\mathbb{Z}_2F$, our argument remains valid because in $\mathbb{Z}_2F$, one has $r - 1 \in \Delta_F \Delta_R$ whenever $r \in R^2$.

3 Proof of the main results

We begin this section with the proof of the Theorem. First of all, we note that the "only if" part of all 3 statements is trivially true, so we proceed with the proof of the "if" parts.

(a) We denote here $c_m = [x_1, \ldots, x_m]$, $m \geqslant 2$. By the conditions of the Theorem and by Lemma 2.2, we have

$$c_n - 1 \in \Delta_F^{n-1}\Delta_R. \tag{4}$$

We are now going to prove by induction on $n \geqslant 2$ that (4) implies $x_1, \ldots, x_n \in R$. When $n = 2$, the argument from the proof of Proposition 2.4 yields the basis of induction.

Let now $n \geqslant 3$. Applying derivation D_n to both sides of (4), we have:

$$1 - x_n^{-1}c_{n-1}x_n = 1 - c_{n-1}c_n \in \Delta_F^{n-2}\Delta_R.$$

This implies $1 - c_{n-1} \in \Delta_F^{n-2}\Delta_R$ because $1 - c_n \in \Delta_F^{n-2}\Delta_R$ by (4). Hence by the inductive assumption, $x_1, \ldots, x_{n-1} \in R$.

Now apply derivation D_{n-1} to both sides of (4): $D_{n-1}(c_{n-1})x_n(1-x_n^{-1}c_n) \in \Delta_F^{n-2}\Delta_R$, so

$$D_{n-1}(c_{n-1})(x_n - 1) \in \Delta_F^{n-2}\Delta_R \tag{5}$$

again because $1 - c_n \in \Delta_F^{n-2}\Delta_R$.

Applying now successively $D_{n-2}, \ldots, D_1$ to both sides of (5), we finally arrive at $(-1)^n x_2x_3 \cdots x_{n-1}(x_n - 1) \in \Delta_R$, so $x_n \in R$.

(b) Denote $u(2m) = [x_1, x_2] \cdots [x_{2m-1}, x_{2m}]$ and proceed by induction on m with Proposition 2.4 and Remark 2.5 as the basis. We have in the group ring $\mathbb{Z}_2F$:

$$[x_1, x_2] \cdots [x_{2m-1}, x_{2m}] - 1 \in \Delta_F\Delta_R. \tag{6}$$

Applying derivations D_{2m-1} and D_{2m} to both sides of (6) yields $x_{2m-1} \in R$ and $x_{2m} \in R$ like in the proof of Proposition 2.4. Thus $[x_{2m-1}, x_{2m}] \in R' \subseteq R^2$, so $u(2(m-1)) \in R^2$, and the proof follows by induction.

(c) Before we proceed with the proof of this part, we have to introduce one special mapping of a group ring. Given a group ring KG of a group G over an arbitrary commutative ring K with the unit, and a subgroup H of G, we can define a mapping $\pi_H \colon KG \to KH$ as follows: if $v = \sum_{g\in G} n_g g$, $n_g \in K$, then $\pi_H(v) = \sum_{g\in H} n_g g$. This mapping has one property we are going to use (see [11] for more details): if $v \in KG$ and $w \in KH$, then $\pi_H(vw) = \pi_H(v)w$. Now denote $u(n,k) = x_1^k \cdots x_n^k$, and let $u(n,k) = c\prod_i r_i^k$ for some $r_i \in R$; $c \in R'$. Apply derivation D_n to both sides of this equality:

$$x_n^{k-1} + \cdots + x_n + 1 = D_n(c) + \sum_i D_n(r_i)(r_i^{k-1} + \cdots + r_i + 1). \tag{7}$$

Now apply the mapping π_R to both sides of (7):

$$\pi_R(x_n^{k-1} + \cdots + x_n + 1) \\ = \pi_R(D_n(c)) + \pi_R\left(\sum_i D_n(r_i)\right)(r_i^{k-1} + \cdots + r_i + 1). \quad (8)$$

The augmentation of the right-hand side of (8) is divisible by k (note that the augmentation of $\pi_R(D_n(c))$ is 0); hence the augmentation of the left-hand side must be also divisible by k which is possible only if $x_n \in R$. Thus $x_n \in R$, so $x_n^k \in R^k$, hence $u(n-1, k) \in R^k$, and the proof follows by induction on n. The theorem is completely proved.

As it was mentioned in the Introduction, our theorem has obvious application to recognizing automorphisms of the free groups F_n. Weakening the "if" part of statement, we can obtain the following

Corollary 3.1. *Let ϕ be an endomorphism of the group $F = F_n$, $n \geqslant 2$, and let u be one of the words $u_1(n)$, $u_2(2m)$, $u_3(n,k)$. Then ϕ is an automorphism if and only if $\phi(u) \in A_F(u)$.*

Proof. Let ϕ take x_i to y_i, $1 \leqslant i \leqslant n$, and let R be the subgroup of F generated by these y_i. Suppose $\phi(u) \in A_F(u)$. This means that $\phi(u) = \alpha(u)$ for some $\alpha \in \operatorname{Aut} F$. Let $z_i = \alpha(x_i)$; then $u(y_1, \ldots, y_n) = u(z_1, \ldots, z_n)$, hence our theorem yields $z_i \in R$, $1 \leqslant i \leqslant n$, so $R = F$. □

Proposition 3.2. *Suppose an element $u \in F_n$ does not satisfy at least one of the following two conditions:*

(i) *u has rank n;*

(ii) *$u \in F'F^k$ for some $k \geqslant 2$.*

Then there is an endomorphism ϕ of F_n which is not an automorphism but $\phi(u) = u$.

Proof. (i) Suppose u has rank less than n. In this case, we can find an automorphism α of F_n such that $\alpha(u) = v(x_1, \ldots, x_m)$, $m < n$. Define now the endomorphism ψ as follows: $\psi(x_i) = x_i$, $1 \leqslant i \leqslant m$; $\psi(x_i) = 1$, $m < i \leqslant n$. Then the endomorphism $\phi = \alpha^{-1}\psi$ fixes u, but ϕ is obviously not an automorphism of F_n.

(ii) The principal idea of the following proof is due to R. M. Bryant.

Suppose u has the form $u = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} c$; $n \geqslant 2$; $\gcd(m_i)_{1 \leqslant i \leqslant n} = 1$; $c \in F'$. Then we have $k_1 m_1 + \cdots + k_n m_n = 1$ for some integers k_i. Define now the endomorphism ϕ as follows: $\phi(x_i) = u^{k_i}$, $1 \leqslant i \leqslant n$. Then $\phi(u) = u$ (note that $\phi(c) = 1$ because the image of ϕ is a cyclic group), but ϕ is clearly not an automorphism. □

An auxiliary result is provided be the following

Proposition 3.3. *Let R be a subgroup of the group F_n, and let $[x_1, \dots, x_n]$ belong to R^2. Then R contains some primitive element of F_n.*

Proof. Let $c_m = [x_1, \dots, x_m]$, $m \geqslant 1$. In the group ring $\mathbb{Z}_2 F$, we have:

$$c_n - 1 \in \Delta_F \Delta_R. \tag{9}$$

Applying the derivation d_n to both sides of (9) yields $1 - c_{n-1}c_n \in \Delta_R$, so $c_{n-1}c_n \in R$ by Lemma 2.3, hence $c_{n-1} \in R$. Applying now a derivation d_j, $2 \leqslant j \leqslant n-1$, to both sides of (9) yields:

$$(1 - c_{j-1}c_j) \prod_{i=j+1}^{n} (x_i - c_i) \in \Delta_R. \tag{10}$$

We are going to prove by "reverse induction" on $j \leqslant n-1$ that (10) implies the following alternative:

(i) $c_{j-1} \in R$, or

(ii) R contains some primitive element of F_n.

We take $j = n-1$ as the basis of induction, and get $(1-c_{n-2}c_{n-1})(x_n - c_n) \in \Delta_R$. Open the brackets: $x_n - c_n - c_{n-2}c_{n-1}x_n + c_{n-2}c_{n-1}c_n \in \Delta_R$. As $c_n \in R$, this implies that either $x_n \in R$ or $c_{n-2}c_{n-1}c_n \in R$. In the latter case we have $c_{n-2} \in R$ as desired.

Proceeding with the step of induction (in case we don't yet have a primitive element in R) and opening the brackets in (10), we see that, as $c_{j+1} \cdots c_n \in R$, we must have at least one of the following elements in R:

(1) element of the form wx_i, $i > j$, with no x_i occurring in w;

(2) element of the form $wx_ic_{i+1} \cdots c_n$, $i > j$, with no x_i occurring in w;

(3) element $c_{j-1}c_j \cdots c_n$.

In the case (1), we clearly have a primitive element in R; in the case (2), we have $c_{i+1} \cdots c_n \in R$ by the inductive assumption, so again $wx_i \in R$. In the case (3), we have $c_{j-1} \in R$ again by the inductive assumption.

The proof of the Proposition is completed in view of $c_1 = x_1$, a primitive element of F_n. □

To conclude this section, we give one more possible application of our theorem:

Corollary 3.4 (cf. [8, I.6], [12])**.** *In a free group F of rank at least n:*

(a) *For an arbitrary $k \geqslant 2$, the element $x_1^k \cdots x_n^k$ is not expressible as a product of less than n k-th powers;*

(b) *For an arbitrary $n = 2m \geqslant 2$, the element $[x_1, x_2] \cdots [x_{n-1}, x_n]$ is not expressible as a product of less than m commutators and less than $(n+1)$ squares.*

Proof. (a) Suppose $x_1^k \cdots x_n^k = y_1^k \cdots y_p^k$, $p < n$. Let R be the subgroup of F generated by $y_1, \ldots, y_p$. By (c) part of out Theorem, we must have then $x_1, \ldots, x_n \in R$, whence R is generated by $p < n$ elements, a contradiction.

(b) The first part is trivial. For the second part, we only have to show that the equality $[x_1], x_2 \cdots [x_{n-1}, x_n] = y_1^2 \cdots y_n^2$ is impossible. Let R be the subgroup of F generated by $y_1, \ldots, y_n$. Were the equality possible, we would have $x_1, \ldots, x_n \in R$ by (b) part of our Theorem. Hence a suitable automorphism of the group F takes $y_1, \ldots, y_n$ to $x_1, \ldots, x_n$, so the right-hand side of the equality is taken to $x_1^2, \ldots, x_n^2$ while the left-hand side is taken to an element of the commutator subgroup under any automorphism of F, a contradiction. □

4 Groups of the form $F/[N', F]$

In this section, we use the method introduced in [6] to prove the following Proposition 4.1. We remind the reader that

$$u_2(2m) = [x_1, x_2] \cdots [x_{2m-1}, x_{2m}].$$

Proposition 4.1. *Let N be a fully invariant subgroup of the free group $F = F_n$ of rank $n = 2m$. Let $G = F/[N', F]$, and suppose an endomorphism ϕ of G satisfies $\phi(u_2(2m)) \in A_G(u_2(2m))$. Then $\phi(G) = G$.*

Proof. We may clearly assume that $\phi(u_2(2m)) = u_2(2m)$. Using the fact that $g - 1 \in \Delta_F \Delta_N \Delta_F$, whenever $g \in [N', F]$, this yields

$$[y_1, y_2] \cdots [y_{2m-1}, y_{2m}] = [x_1, x_2] \cdots [x_{2m-1}, x_{2m}] \bmod \Delta_F \Delta_N \Delta_F \qquad (11)$$

with $y_i = \phi(x_i)$. Taking Fox derivatives $D_1, \ldots, D_n$ of both sides of (11), we get n congruences modulo $\Delta_N \Delta_F$. After that, taking each of the derivatives $d_1, \ldots, d_n$ of both sides of each of n congruences obtained, we arrive at a system of n^2 congruences modulo Δ_N which can be written in the matrix form as $J_\phi A = B$, where $J_\phi = \|D_i(y_j)\|$ is the "Jacobian matrix" of ϕ, A is some matrix that is not essential for our purposes, and B has the form

$$\begin{pmatrix} B_1 & & & \\ & B_2 & & * \\ 0 & & \cdots & \\ & & & B_m \end{pmatrix},$$

where B_k is 2×2 matrix

$$\begin{pmatrix} x_{2k-1}^{-1} x_{2k}^{-1}(x_{2k} - 1) & 1 - x_{2k-1}^{-1} x_{2k}^{-1}(x_{2k-1} - 1) \\ -x_{2k}^{-1} & x_{2k}^{-1} - x_{2k}^{-1} x_{2k-1} \end{pmatrix},$$

$1 \leqslant k \leqslant m$.

It can be checked easily that every matrix B_k is invertible over $\mathbb{Z}F$, and hence over $\mathbb{Z}F$ (mod Δ_N). This implies that the matrix B, and hence also the Jacobian matrix J_ϕ is invertible over $\mathbb{Z}F \pmod{\Delta_N}$. Now this implies, by a result of Krasnikov [7], that $y_1, \ldots, y_{2m}$ generate F_n modulo N'. It follows that $y_1, \ldots, y_{2m}$ also generate F_n modulo any normal subgroup H of N with N/H nilpotent, in particular modulo $[N', F]$. This completes the proof. □

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