Open problems in combinatorial group theory. Second edition

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Contents

Introduction

This is a collection of open problems in combinatorial group theory, which is based on a similar list available online at our web site
http://www.grouptheory.org

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In three years since the first edition of this collection was published in [18], over 40 new problems have been added, and, more importantly, 20 problems have been solved. We hope that this is an indication of our work being useful. In the present edition we have added two new sections: on Groups of Matrices and on Growth.

In selecting the problems, our choices have been, in part, determined by our own tastes. In view of this, we have welcomed suggestions from other members of the community. We want to emphasize that many people have contributed to the present collection, especially to the Background part. In particular, we are grateful to G. Bergman, G. Conner, W. Dicks, R. Gilman, I. Kapovich, V. Remeslennikov, V. Roman'kov, E. Ventura and D.Wise for useful comments and discussions. Still, we admit that the background we currently have is far from complete, and it is our ongoing effort to make it as complete and comprehensive as possible. We invite an interested reader to check on our online list for a latest update.

One more thing concerning our policy that we would like to point out here, is that we have decided to keep on our list those problems that have been solved after the first draft of the list was put online in *June 1997*, since we believe those problems are an important part of the list anyway, because of their connections to other, yet unsolved, problems. Solved problems are marked by a $*$, and a reference to the solution is provided in the background.

We have arranged the problems under the following headings:
Outstanding Problems, Free Groups, One-relator Groups, Finitely Presented Groups, Hyperbolic and Automatic Groups, Braid Groups, Nilpotent Groups, Metabelian Groups, Solvable Groups, Groups of Matrices, Growth.

Disclaimer. We want to emphasize that all references we give and attributions we make reflect our personal opinion based on the information we have. In particular, if we are aware that a problem was raised by a specific person, we make mention of that here. We welcome any additional information and/or corrections on these issues.

1 Outstanding Problems

(O1) The Andrews-Curtis conjecture. *Let $F = F_n$ be the free group of a finite rank $n \geq 2$ with a fixed set $X = \{x_1, ..., x_n\}$ of free generators. A set $Y = \{y_1, ..., y_n\}$ of elements of F generates the group F as a normal subgroup if and only if Y is Andrews-Curtis equivalent to X, which means one can get from X to Y by a sequence of Nielsen transformations together with conjugations by elements of F.*

This problem is of interest in topology as well as in group theory. A topological interpretation of this conjecture was given in the original paper by

J. Andrews and M. Curtis [3]. A more interesting topological interpretation arises when one allows one more transformation, "stabilization", when Y is extended to $\{y_1, ..., y_m, x_\nu\}$, where x_ν is a new free generator (i.e., $y_1, ..., y_m$ do not depend on x_ν), and the converse of this transformation. Then the Andrews-Curtis conjecture is equivalent to the following (see [184]): two contractible 2-dimensional polyhedra P and Q can both be embedded in a 3-dimensional polyhedron S so that S geometrically contracts to P and Q. Note that this is true if 3 is replaced by 4 – this follows from a result of Whitehead. The problem is amazingly resistant; very few partial results are known. A good group-theoretical survey is [32]. For a topological survey, we refer to [80].

The prevalent opinion is that the conjecture is false; however, not many potential counterexamples are known. Two of them are given in the survey [32] by R. Burns and O. Macedonska; a one-parameter family of potential counterexamples appears in [1]. Recently, a rather general series of potential counterexamples in rank 2 was reported in [142]. Further potential counterexamples are given in [146].

It might be of interest that, by using genetic algorithms, A. D. Myasnikov and A. G. Myasnikov [145] showed that all presentations of the trivial group with the total length of relators up to 12 satisfy the Andrews-Curtis conjecture.

Finally, we mention a positive solution of a similar problem for *free solvable groups* by A. G. Myasnikov [144].

(O2) The Burnside problem. *For what values of n are all groups of exponent n locally finite? Of particular interest are $n = 5$, $n = 8$, $n = 9$ and $n = 12$ – values for which, by the experts' opinion, groups of exponent n have a remote chance of being locally finite.*

In contrast with the previous problem (O1), the bibliography on the Burnside problem consists of several hundred papers. We only mention here that E. Golod [64] constructed the first example of a periodic group which is not locally finite; his group however does not have bounded exponent. The first example of an infinite finitely generated group of bounded exponent is due to P. S. Novikov and S. I. Adian [154]. We refer to the book [156] for a survey on results up to 1988, and to the papers [87], [126] for treatment of the most difficult case where the exponent is a power of 2.

We also note that recent work of J. McCammond [133] offers a new approach to the Burnside problem via (generalized) small cancellation theory.

(O3) Whitehead's asphericity problem. *Is every subcomplex of an aspherical 2-complex aspherical? Or, equivalently: if $G = F/R =< x_1, ..., x_n | r_1, ..., r_m, ... >$ is an aspherical presentation of a group G (i.e., the corresponding relation module $R/[R, R]$ is a free ZG-module), is every presentation of the form $\langle x_1, ..., x_n | \ r_{i_1}, ..., r_{i_k}, ...\rangle$, aspherical as well ?*

This problem has received considerable attention in the '80s. We mention here a paper by J. Huebschmann [84] that contains a wealth of examples of 2-complexes for which Whitehead's asphericity problem has a positive solution. J. Howie [81] points out a connection between Whitehead's problem and some other problems in low-dimensional topology (e.g. the Andrews-Curtis conjecture). We refer to [123] for more bibliography on this problem. Among more recent papers, we mention a paper by E. Luft [121], where he gives a rather elementary self-contained proof of the main result of Howie's paper (see above) and strengthens the result at the same time.

Recently, S. V. Ivanov [90] discovered a connection between Whitehead's asphericity problem and the problem (O12)(a) below. A somewhat more general result was later obtained by I. Leary [112].

(O4) *The isomorphism problem for one-relator groups.*

Z. Sela [168] has solved the isomorphism problem for torsion-free hyperbolic groups that do not split (as an amalgamated product or an HNN extension) over the trivial or the infinite cyclic group. It is not known however which one-relator groups are hyperbolic (cf. problem (O6)). Earlier partial results are [160] and [161]. We also note that two one-relator groups with relators r_1 and r_2 being isomorphic does not imply that r_1 and r_2 are conjugate by an automorphism of a free group, which deprives one from the most straightforward way of attacking this problem; see [138].

(O5) *The conjugacy problem for one-relator groups.*

The conjugacy problem for one-relator groups with torsion was solved by B. B. Newman [152]. Some other partial results are known; see e.g. [124] for a survey.

(O6) *Is every one-relator group without Baumslag-Solitar subgroups hyperbolic?*

Note that every one-relator group with torsion is hyperbolic since the word problem for such a group can be solved by Dehn's algorithm – see the paper by B. B. Newman cited in the background to (O5). Therefore, it suffices to consider torsion-free one-relator groups. We also note that the following weak form of this problem was answered in the affirmative. A group G is called a CSA group if every maximal abelian subgroup M of G is malnormal, i.e., for any element g in G, but not in M, one has $M^g \cap M = \{1\}$. It is known that every torsion-free hyperbolic group is CSA. Now the following weak form of (O6) holds: A torsion-free one-relator group is CSA if and only if it does not contain metabelian Baumslag-Solitar groups $BS(1,p)$ and subgroups isomorpfic to $F_2 \times Z$ [61].

There are also several partial (positive) results on this problem in [91].

(O7) *Let G be the direct product of two copies of the free group F_n, $n \geq 2$, generated by $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_n\}$, respectively. Is it true that every generating system of cardinality $2n$ of the group G is Nielsen equivalent to $\{x_1, ..., x_n, y_1, ..., y_n\}$?*

The importance of this problem is in its relation to two outstanding problems in low-dimensional topology, to the Poincaré and Andrews-Curtis conjectures – see e.g. the survey [68] for details.

***(O8) Tarski's problems.** *Let $F = F_n$ be the free group of rank n, $Th(F)$ the elementary theory of F, i.e., all sentences in the language of group theory which are true in F.*
(a) *Is it true that $Th(F_2) = Th(F_3)$?*
(b) *Is $Th(F)$ decidable?*

(a) By a result of Yu. Merzlyakov [140], all free groups of finite rank $n \geq 2$ satisfy the same positive sentences. G. Sacerdote [167] re-proved this result, and also proved that all free groups of finite rank $n \geq 2$ satisfy the same $(\forall\exists)$ and the same $(\exists\forall)$ sentences.

(b) Several fragments of the elementary theory of a free group of finite rank were shown to be decidable. We mention here important work of G. Makanin [129] and A. Razborov [162] on solving equations and systems of equations in a free group. G. Makanin [130] also proved decidability of the universal and positive theories of a free group.

A complete positive solution of both Tarski's problems was announced by O.Kharlampovich and A.Myasnikov in [102]. A subsequent series of papers [100], [103], [104], [105] contains a complete exposition of their method with full proofs. Preprints are available at
http://www.math.mcgill.ca/˜ olga/publications.html

Recently, a positive solution of the problem (O8)(a) was also announced by Z.Sela. His 6 preprints under a common title of *Diophanite Geometry over Groups* are available at *http://www.ma.huji.ac.il/˜ zlil/*

(O9) The Hanna Neumann conjecture. *If H and K are non-trivial subgroups of a free group, then*
$rank(H \cap K) - 1 \leq (rank(H) - 1)(rank(K) - 1)$.

It is convenient to write $(rank - n)(H) = max(rank(H) - n, 0)$.

Hanna Neumann proved that $(rank-1)(H \cap K) \leq 2(rank-1)(H)(rank-1)(K)$ and conjectured that the coefficient 2 could be removed. R. Burns [31] showed $(rank - 1)(H \cap K) \leq (rank - 1)(H)(rank - 1)(K) + max((rank - 1)(H)(rank - 2)(K), (rank - 2)(H)(rank - 1)(K))$, thus proving the conjectured inequality when both subgroups have rank two. W. Neumann [150] formulated a stronger version of the Hanna Neumann conjecture, and proved the stronger version of Burns' bound. All subsequent results have applied to the stronger version.

G. Tardos [177] proved the conjectured inequality when one subgroup has rank two. W. Dicks [41] translated the stronger version into a graph-theoretic conjecture.

G. Tardos [178] improved Burns' bound showing $(rank-1)(H \cap K) \leq (rank-1)(H)(rank-1)(K)+max((rank-2)(H)(rank-2)(K)-1,0)$, thus proving the conjectured inequality when both subgroups have rank three.

W. Dicks and E. Formanek [43] improved Tardos' bound showing $(rank-1)(H \cap K) \leq (rank-1)(H)(rank-1)(K)+(rank-3)(H)(rank-3)(K)$, thus proving the conjectured inequality when one subgroup has rank three.

B. Khan [98] showed that if one of the subgroups, say H, has a generating set consisting of only positive words, then H is not part of any counterexample to the conjecture. A similar result was independently obtained by J. Meakin and P. Weil [139].

A complete proof of the Hanna Neumann conjecture was claimed by W. S. Jassim [93], but W. Dicks [42] has given an example which he feels makes it appear likely that the argument is not valid, although Jassim is at this time not in agreement.

***(O10)** *Is the automorphism group of the free group of rank 2 linear? Or, equivalently, is the braid group B_4 linear ?*

E. Formanek and C. Procesi [55] proved that the automorphism group of a free group of rank n is not linear if $n \geq 3$. The "Or, equivalently" statement is due to J. Dyer, E. Formanek and E. Grossman [49].

The problem has been settled in the affirmative by D. Krammer [108]. Later on, S. Bigelow [25] and D. Krammer [109] proved that the Krammer representation of the braid group B_n is faithful for every n, and therefore all braid groups are linear.

(O11) *Is there an infinite finitely presented periodic group?*

Needless to say, infinite periodic groups constructed by Golod, Novikov-Adian, and Olshanskii (see the background to (O2)) are infinitely related.

(O12) **(a)** (I. Kaplansky) *Can the (integral) group ring of a torsion-free group have zero divisors?*
(b) Is it true that the group of units of a group algebra kG, where G is torsion-free and k is a field, is generated by k^* and G ?

The bibliography on these problems consists of over a hundred papers. One of the highest points here is a result of P. Kropholler, P. Linnell and J. Moody [110] which implies, in particular, that the integral group ring of a torsion-free virtually solvable group has no zero divisors. We refer to [159] for a survey on results up to 1977.

More recently, T. Delzant [40] has shown that group rings of a large class of torsion-free hyperbolic groups have no zero divisors.

We also note that S. Ivanov [90] discovered a connection between the problem (O12)(a) and Whitehead's asphericity problem (O3).

2 Free Groups

These are problems about free groups, their automorphisms and related issues. See also problems (O1), (O7), (O8), (O9), (O10).

(F1) (a) *Is there an algorithm for deciding if a given automorphism of a free group has a non-trivial fixed point ?*
(b) *Is there an algorithm for deciding if a given finitely generated subgroup of a free group is the fixed point group of some automorphism ?*

S. Gersten [57], [58] proved that the fixed point group $Fix(\phi)$ of any automorphism ϕ of a free group F_n of finite rank is finitely generated. A simpler proof was given by D. Cooper [37], and R. Goldstein and E. Turner [63] obtained a similar result for arbitrary endomorphisms of a free group. M. Bestvina and M. Handel [23] showed that the rank of $Fix(\phi)$ cannot exceed n. In [86], this was generalized to arbitrary endomorphisms.

All these results however do not give an effective procedure for detecting fixed points of a given automorphism. M. Cohen and M. Lustig [35] obtained several useful partial results and, in particular, solved the problem for *positive* automorphisms (i.e., for those that take every free generator to a positive word).

For part (b), we just note that a subgroup of rank $> n$ cannot possibly be the fixed point group of an automorphism by the result of Bestvina and Handel mentioned above. On the other hand, any cyclic subgroup generated by an element u which is not a proper power, is the fixed point group of the inner automorphism induced by u.

***(F2)** (H. Bass) *Does the automorphism group of a free group satisfy the "Tits alternative" ?*

Yes, it does – see [22].

***(F3)** (V. Shpilrain) *If an endomorphism ϕ of a free group F of finite rank takes every primitive element to another primitive, is ϕ an automorphism of F?*

This problem was solved in the affirmative for $n = 2$ by V. Shpilrain [172] and by S. Ivanov [89]. S. Ivanov also showed that the answer is positive in the general case under an additional assumption on ϕ to have a *primitive pair* in the image.

Recently, D. Lee [114] has settled the problem completely, for every n.

***(F4)** Denote by $Orb_\phi(u)$ the orbit of an element u of the free group F_n under the action of an automorphism $\phi \in Aut(F_n)$. That is, $Orb_\phi(u) =$

$\{v \in F_n,\ v = \phi^m(u)\ for\ some\ m \geq 0\}$. If an orbit like that is finite, how many elements can it possibly have if u runs through the whole group F_n, and ϕ runs through the whole group $Aut(F_n)$?

There is a nice simple argument showing that the number of elements in an orbit is bounded by a function depending only on n – see [115]. Suppose that for some automorphism ϕ of $F = F_n$, we have $\phi^k(g) = g$ and $\phi^l(g) \neq g$ for $0 < l < k$. Consider the action of ϕ on the subgroup $H = Fix(\phi^k)$ consisting of all elements fixed by ϕ^k. (This subgroup is clearly invariant under ϕ.) Then ϕ has order k as an element of $Aut(H)$. Since H has rank at most n by [23], this gives a bound for k in terms of n, since there is a bound for the order of a torsion element in $GL_n(Z)$, hence also for the order of a torsion element in $Aut(F_n)$ because the kernel of the map from $Aut(F_n)$ to $GL_n(Z)$ is torsion-free.

Recently, A. Myasnikov and V. Shpilrain [148] showed that the converse is also true, and therefore, in the free group F_n, there is an orbit $Orb_\phi(u)$ of cardinality k if and only if there is an element of order k in the group $Aut(F_n)$. Their preprint is available at
http://zebra.sci.ccny.cuny.edu/web/shpil/orbit.ps

Note that possible orders of torsion elements of the group $Aut(F_n)$ were described in [134] and [106].

(F5) (H. Bass) *Is the automorphism group of a free group "rigid", i.e., does it have only finitely many irreducible complex representations in every dimension?*

S. Humphries has shown recently that braid groups are not rigid. This implies that the automorphism group $Aut(F_2)$ is not rigid. The preprint is available at *http://zebra.sci.ccny.cuny.edu/web/problems/humphrie.ps*

(F6) *The conjugacy problem for the automorphism group of a free group of finite rank.*

An outer automorphism Φ of a free group F of finite rank is said to be reducible if there is a free factorization $F = F_1 \star \cdots \star F_k \star F'$ such that Φ permutes the conjugacy classes of the subgroups $F_1, \cdots, F_k$; otherwise, Φ is irreducible. Z. Sela [168] and J. Los [120] obtained algorithms which decide if two irreducible outer automorphisms are conjugate in the group of outer automorphisms of F.

***(F7)** (V. Shpilrain) *Denote by $Epi(n,k)$ the set of all homomorphisms from a free group F_n onto a free group F_k; $n, k \geq 2$. Are there 2 elements $g_1, g_2 \in F_n$ with the following property: whenever $\phi(g_i) = \psi(g_i),\ i = 1, 2$, for some homomorphisms $\phi, \psi \in Epi(n,k)$, then $\phi = \psi$? (In other words, every homomorphism from $Epi(n,k)$ is completely determined by its values on just 2 elements.)*

S. Ivanov [88] has proved that every *injective* homomorphism from $Epi(n,k)$ is completely determined by its values on just 2 elements.

Recently, D. Lee [113] has settled the problem completely.

(F8) (W. Dicks, E. Ventura) *Let ϕ be an endomorphism of a free group F_n, and S a subgroup of F_n having finite rank. Is it true that $rank(Fix(\phi) \cap S) \leq rank(S)$?*

This is true if $S = F_n$, although the only known proof of this fact is highly non-trivial (see [23] for the case where ϕ is an automorphism, and [86] for an extension of this result to arbitrary endomorphisms). For S an arbitrary finite rank subgroup of F_n, the result was established in the case where ϕ is injective [45].

(F9) (A.I.Kostrikin) *Let F be the free group of rank 2 generated by x, y. Is the commutator $[x, y, y, y, y, y, y]$ a product of fifth powers in F? (If not, then the Burnside group $B(2, 5)$ is infinite.)*

***(F10)** (A.I.Mal'cev) *Can one describe the commutator subgroup of a free group by a first order formula in the language of group theory ?*

The negative answer to this problem follows from a positive solution of Tarskii's problem (O8)(b) by O.Kharlampovich and A.Myasnikov (see the background to problem (O8)).

Indeed, if the answer to the problem (F10) was positive, this would imply that the elementary theory of a free non-abelian group F (with constants from F in the language) is undecidable, since there is no algorithm for deciding if a given equation in a free group F has solutions from $[F, F]$ [48].

(F11) (G. Bergman) *Let S be a subgroup of a free group F, and R a retract of F. Is it true that the intersection of R and S is a retract of S?*

We can only remark here that the intersection of two retracts of a free group is itself a retract, but a proof of this fact is much harder than one would expect – see [20].

(F12) (G. Baumslag) *Let $F = F_n$ be a free group generated by $\{x_1, ..., x_n\}$, and let F^Q be the free Q-group, i.e., the free object of rank n in the category of uniquely divisible groups. Consider the map $x_i \longrightarrow (1 + x_i)$ from the generators of F^Q into the formal power series ring $Q\langle\langle x_1, ..., x_n\rangle\rangle$ with coefficients in Q. It is known that this map induces a homomorphism $\lambda : F^Q \longrightarrow Q\langle\langle x_1, ..., x_n\rangle\rangle$ (the Magnus homomorphism). Is λ injective? Or, equivalently, is the group F^Q residually torsion-free nilpotent?*

A construction of the group F^Q in terms of free products with amalgamation is given in [10].

The best known result about the Magnus homomorphism of the group F^Q is due to G. Baumslag [11]. He proved that the Magnus homomorphism is one-to-one on any subgroup of F^Q of the form $\langle F, t | u = t^n \rangle$.

This problem can be re-formulated in a more general form, where the ring Q of rationals is replaced by some other associative ring A. In [147], it was shown how to construct a free group F^A for an arbitrary unitary associative ring A of characteristic 0. In particular, if $A = Z[X]$ is a ring of polynomials with integral coefficients, then F^A is Lyndon's free group.

In [56], it was shown that the Magnus homomorphism of $F^{Z[x]}$ into the corresponding power series ring is an emebedding. Moreover, the Magnus homomorphism is an embedding for *every* unitary associative ring A of characteristic 0 if and only if it is an embedding in the case where $A = Q$.

(F13) (I. Kapovich) *Is the group F^Q in the previous problem linear?*

We note that Lyndon's free $Z[x]$-group $F^{Z[x]}$ (see the background to (F12)) is linear. Indeed, the group $F^{Z[x]}$ is discriminated by F [122], hence it is universally equivalent to F, therefore it is embeddable into an ultrapower of F, which is linear.

(F14) *Let F be a non-cyclic free group of finite rank, and G a finitely generated residually finite group. Is G isomorphic to F if it has the same set of finite homomorphic images as F does?*

We note that the answer is "yes" for a *free metabelian* group of finite rank – see [153].

(F15) (V. Shpilrain) *Let F be a non-cyclic free group, and R a non-cyclic subgroup of F. Suppose that the commutator subgroup $[R, R]$ is a normal subgroup of F. Is R necessarily a normal subgroup of F?*

This question was motivated by the following result of [5]: if R and S are normal subgroups of F, and $[R, R] \subseteq [S, S]$, then $R \subseteq S$. M. Dunwoody [46] showed that the condition on R being normal cannot be dropped, but it is not known whether or not the condition on S being normal can be dropped.

***(F16)** (V. Remeslennikov) *Let R be the normal closure of an element r in a free group F with the natural length function, and suppose that s is an element of minimal length in R. Is it true that s is conjugate to one of the following elements: $r, r^{-1}, [r, f]$, or $[r^{-1}, f]$ for some element f ?*

This question was motivated by a well-known result of Magnus (see e.g. [124]): if two elements, r and s, of a free group F have the same normal closure in F, then s is conjugate to r or r^{-1}.

The negative answer has been recently given by J.McCool in [136]. His preprint is available at *http://zebra.sci.ccny.cuny.edu/web/problems/mccool.ps*

(F17) (M.Wicks) *Let F_n be a non-cyclic free group of rank n, and $P(n,k)$ the number of its primitive elements of length k. What is the growth of $P(n,k)$ as a function of k, with n fixed ?*

We note that the function $P(n,k)$ is recursive, i.e., its values can be actually computed.

It is shown in [27] that for some constants c_1, c_2, one has $c_1 \cdot (2n-3)^k \leq P(n,k) \leq c_2 \cdot (2n-2)^k$ if $n \geq 3$. The preprint is available at *http://zebra.sci.ccny.cuny.edu/web/shpil/measurefin5.ps*

For $n = 2$, the precise number of *cyclically reduced* primitive elements of length k is given in [148]. The preprint is available at *http://zebra.sci.ccny.cuny.edu/web/shpil/orbit.ps*

(F18) (C. Sims) *Is the c-th term of the lower central series of a free group of finite rank the normal closure of basic commutators of weight c ?*

This is known to be true for $c \leq 5$.

***(F19)** (A. Gaglione, D. Spellman) *Let F be a non-cyclic free group, and G the Cartesian (unrestricted) product of countably many copies of F. Is the group $G/[G,G]$ torsion-free?*

There is a 2-torsion in this group – see [103].

***(F20)** (L. Comerford) *If an equation over a free group F has no solutions in F, is there a finite quotient of F in which the equation has no solutions? (If so, this provides another proof of Makanin's theorem).*

The answer was shown to be negative in [38].

(F21) (P. M. Neumann) *Let G be a free product amalgamating proper subgroups H and K of A and B, respectively. Suppose that A, B, H, K are free groups of finite ranks. Can G be simple?*

It cannot if H and K are of infinite index – see [33].

Recently, S. V. Ivanov and P. Schupp [92] have strengthened this result by showing that the answer is still negative if either one of the subgroups H, K has infinite index in A or B, respectively.

We also note that the affirmative solution of this problem was announced in [30].

(F22) (A. Olshanskii) *Does the free group of rank 2 have an infinite ascending chain of fully invariant subgroups, each being generated (as a fully invariant subgroup) by a single element?*

(F23) (A. Myasnikov, V. Remeslennikov) *Let G be a free product of two isomorphic free groups of finite ranks amalgamated over a finitely generated subgroup.*

*(**a**) *Is the conjugacy problem solvable in G?*
(b) *Is there an algorithm to decide if G is free ?*
(c) *Is there an algorithm to decide if G is hyperbolic?*

If the amalgamated subgroup is cyclic then the first two problems have affirmative answers:

(a) is due to S. Lipschutz [117]. See also [118], and

(b) is due to Whitehead since a one-relator group is free if and only if the relator is part of a basis of the ambient free group.

It has been brought to our attention by C. F. Miller that a slight adjustment of the argument in Theorem 10 of [141] shows that there are free products of two free groups of the same finite rank with finitely generated amalgamated subgroups, that have unsolvable conjugacy problem.

(F24) (G. Baumslag, A. Myasnikov, V. Remeslennikov) *Is a free product of two equationally noetherian groups equationally noetherian? (A group is called equationally noetherian if every system of equations in finitely many variables in this group is equivalent to a finite subsystem.)*

R. Bryant [29] and V. Guba [71] proved that free groups are equationally noetherian. See also [175].

For a general discussion on this and related problems, we refer to [16].

(F25) (A. Myasnikov, V. Shpilrain) *Let u be an element of a free group F_n, whose length $|u|$ cannot be decreased by any automorphism of F_n. Let $A(u)$ denote the set of elements $\{v \in F_n;\ |v| = |u|,\ f(v) = u$ for some $f \in Aut(F_n)\}$.*
(a) *Is it true that the cardinality of $A(u)$ is bounded by a polynomial function of $|u|$?*
(b) *If the free group has rank 2, is it true that the cardinality of $A(u)$ is bounded by $c \cdot |u|^2$ for some constant c, which is independent of u ?*

This question was motivated by complexity issues for Whitehead's algorithm that determines whether or not a given element of a free group of finite rank is an automorphic image of another given element. It is known that the first part of this algorithm (reducing a given free word to a free word of minimal possible length by "elementary" Whitehead automorphisms) is pretty fast (of quadratic time with respect to the length of the word). On the other hand, the second part of the algorithm (applied to two words of the same minimal length) was always considered very slow. In fact, the procedure outlined in the original paper by Whitehead suggested this part of the algorithm to be of superexponential time with respect to the length of the words.

However, a standard trick in graph theory shows that there is an algorithm of at most exponential time. Whether or not this algorithm is actually of polynomial time, is unknown. The affirmative answer to the problem (F25)(a) would imply that indeed it is.

Recently, A. Myasnikov and V. Shpilrain [148] showed that the answer to (F25)(a) is affirmative for the free group of rank 2. The preprint is available at *http://zebra.sci.ccny.cuny.edu/web/shpil/orbit.ps*

We note that computer experiments suggest that the answer to (F25)(b) is affirmative as well, with $c = 8$.

(F26) (M. Bestvina) *Let ϕ, ψ be two automorphisms of a free group F_n. Is it true that the intersection of $Fix(\phi)$ and $Fix(\psi)$ equals $Fix(\alpha)$ for some automorphism α of F_n ?*

The answer is known to be positive for the free group of rank 2 – see [181].

In a free group of arbitrary finite rank, the intersection of $Fix(\phi)$ and $Fix(\psi)$ is always a *free factor* of some $Fix(\alpha)$ – see [132].

(F27) (W. Magnus) *Let u be an element of a free group F_n. An element r in F_n is called a normal root of u if u belongs to the normal closure of r in the group F_n. Can an element u, which does not belong to the commutator subgroup $[F_n, F_n]$, have infinitely many non-conjugate normal roots ?*

Magnus himself considered some special cases – see [127]. In particular, he showed that if u is primitive, then, up to conjugacy and inversion, the only normal root of u is u itself, whereas if $u = [x, y]$, then, apart from conjugates of u and its inverse, normal roots of u are just the primitive elements of F_2. Magnus also found the normal roots of $x^p y^p$ whenever p is a prime, and Steinberg [176] extended this to finding all roots of $x^p y^q$ whenever p, q are primes.

Recently, J. McCool [135] showed that if u is of the form $x^k y^l$, then u has only finitely many normal roots, and those can be found algorithmically. He also gives a description of the set of normal roots of any element of the form $[x^k, y]$.

(F28) (S. Sidki) *Let S be a subgroup of index 2 in the group F_2, and let R be an isomorphic copy of S (in F_2). Denote by f an isomorphism between S and R. Is there necessarily a non-trivial subgroup H in S which is invariant under f ?*

For a general setup that motivated this problem, we refer to a recent preprint [149], which is available at
http://zebra.sci.ccny.cuny.edu/web/problems/sidki.ps

(F29) (W. Dicks, E. Ventura) *Let H be a subgroup of a free group F_n, and let $r(H)$ denote the rank of H. We call H inert if $r(H \cap K)$ is not bigger than $r(K)$ for any subgroup K of F_n. Is every retract of F_n inert?*

Compare to the problems (F8) and (F11).

(F30) (W. Dicks, E. Ventura) *Let H be a subgroup of a free group F_n, and let $r(H)$ denote the rank of H. We call H compressed if $r(H)$ is not bigger than $r(K)$ for any subgroup K containing H. If H is compressed in F_n, is H necessarily inert ? (See the previous problem (F29).)*

Clearly, if a subgroup of F_n is inert (see problem (F29)), then it is compressed in F_n. By the Nielsen-Schreier formula, the two notions coincide for subgroups of finite index in F_n.

Note also that, since every retract of F_n is compressed, the affirmative answer to this problem would imply the affirmative answer to the problem (F29).

(F31) (J. Stallings) *The equalizer of two homomorphisms $\alpha,\ \beta : F_n \to F_m$ is the group $Eq(\alpha, \beta) = \{x \in F_n : \ \alpha(x) = \beta(x)\}$. Is it true that if α is injective, then the rank of $Eq(\alpha, \beta))$ is at most n ?*

In [174], Stallings notes that there are two homomorphisms from a free group of rank 2 to a free group of rank 1, whose equalizer is not finitely generated.

However, if both $m, n \geq 2$, then the equalizer $Eq(\alpha, \beta))$ is finitely generated – this was proved in [63].

In the case where α is injective and β can be lifted to an injective endomorphism of F_m, the rank of $Eq(\alpha, \beta))$ is indeed bounded by n – see [45].

Finally, we note that G. Bergman showed in [20] that if there is a map $\gamma : F_m \to F_n$ such that $\gamma\alpha$ and $\gamma\beta$ are both the identity on F_n, then $Eq(\alpha, \beta)$ is an intersection of free factors of F_n. In particular, the rank of $Eq(\alpha, \beta)$ is bounded by n in that case.

(F32) *An automorphism of a free group F is called an IA-automorphism if it is Identical on the Abelianization $F/[F, F]$. Obviously, all IA-automorphisms form a (normal) subgroup $IA(F)$ of the group $Aut(F)$. Is the group $IA(F_n)$ finitely presented for $n > 3$?*

By a result of Magnus, the group $IA(F_n)$ is finitely generated for every n. By a classical result of Nielsen, $IA(F_2)$ is isomorphic to F_2 and is therefore finitely presented (see the discussion after Proposition I.4.5 in [124]).

J. McCool and S. Krstic [137] proved that the group $IA(F_3)$ is NOT finitely presented. The problem remains open for $n > 3$.

(F33) (A. Casson) *Let ϕ be an automorphism of F_n. Is it true that there is a subgroup K of finite index in F_n, invariant under ϕ, such that every eigenvalue of ϕ_K is equal to a root of 1, where ϕ_K denotes the induced automorphism of $K/[K, K]$?*

(F34) (A. Myasnikov, V. Shpilrain) *Let F_n be the free group of a finite rank n, with generators $x_1, ..., x_n$. An element u of F_n is called* positive *if no x_i*

occurs in u to a negative exponent. An element u is called potentially positive *if $\alpha(u)$ is positive for some automorphism α of the group F_n. Finally, u is called* stably potentially positive *if it is potentially positive as an element of F_m for some $m \geq n$.*
(a) *Is the property of being potentially positive algorithmically recognizable?*
(b) *Are there stably potentially positive elements of F_n that are not potentially positive ?*

Originally, these problems were motivated by recent results of B. Khan [98] and J. Meakin and P. Weil [139], who established the Hanna Neumann conjecture (see Problem (O9)) in the case where one of the subgroups has a positive generating set.

Clearly, the cited result remains valid upon replacing "positive" with "potentially positive" or even with "stably potentially positive".

3 One-relator Groups

(OR1) (G. Baumslag) *Are all one-relator groups with torsion residually finite?*

For a background to this problem, see the survey [12].

Several partial (positive) results were recently obtained by D. Wise [183]. His preprint is available at *http://www.math.mcgill.ca/wise/papers.html*

(OR2) Is the isomorphism problem solvable for one-relator groups with torsion?

See the background to the problem (O4).

(OR3) (A. Myasnikov) *Is the complexity of the word problem for every one-relator group quadratic, i.e., is there for every one-relator group an algorithm solving the word problem in quadratic time with respect to the length of a word? In polynomial time?*

(OR4) *Is the generalized word problem solvable for one-relator groups? That is, is there an algorithm for deciding if a given element of the group belongs to a given finitely generated subgroup?*

(OR5) *Is it true that if the relation module of a group G is cyclic, then G is a one-relator group?*

J. Harlander [77] showed that the answer is "yes" in the case where G is finitely generated and solvable.

(OR6) (G. Baumslag) *Let $H = F/R$ be a one-relator group, where R is the normal closure of an element $r \in F$. Then, let $G = F/S$ be another one-relator group, where S is the normal closure of $s = r^k$ for some integer k. Is G residually finite whenever H is ?*

See the survey [12].

(OR7) (G. Baumslag) *Let $G = F/R$ be a one-relator group with the relator from $[F, F]$.*
(a) *Is G hopfian ?*
***(b)** *Is G residually finite ?*
***(c)** *Is G automatic ?*

A solution of the problems (b) and (c) was communicated to us by A.Olshanskii. In fact, the commutator subgroup $[F, F]$ can be replaced here by *any* non-cyclic subgroup of a free group F; the answer will still be negative. It follows from a result of A. Olshanskii [155] that for any m, every non-cyclic subgroup H of F contains a subgroup K, which is a free group of rank m, with the following property: for any normal subgroup U of K, the intersection of K and the normal closure of U in F is again U.

To apply this result to our situation, take two elements, x and y, that generate a subgroup $K = F_2$ of $H = [F, F]$ with the property described above. Let r be a Baumslag-Solitar relator built on these two elements; for example, take $r = xyx^{-1}y^{-2}$. Let U be the normal closure (in K) of r. Then, from what is said in the previous paragraph, it follows that the normal closure of U in F (call it V) intersects K in U. Therefore, the (one-relator) group F/V contains a subgroup KV/V which is isomorphic to a Baumslag-Solitar group, hence F/V can be neither residually finite nor automatic.

(OR8) (G. Baumslag) *The same as (OR7), but for a relator of the form $[u, v]$.*

This problem, as well as (OR7)(a), is motivated by the desire to find a non-hopfian one-relator group which is essentially different from any of the Baumslag-Solitar groups [19].

(OR9) (D.Moldavanskii) *Are two one-relator groups isomorphic if each of them is a homomorphic image of the other?*

(OR10) *Is every one-relator group without non-abelian metabelian subgroups, automatic?*

Note that hyperbolic groups are automatic, and, in particular, an amalgamated product of two free groups with finitely generated subgroups amalgamated is hyperbolic if at least one of the subgroups is malnormal [101].

Furthermore, an amalgamated product of two finitely generated abelian groups is automatic [14].

(OR11) (C.Y.Tang) *Are all one-relator groups with torsion conjugacy separable?*

(OR12) *Are all freely indecomposable one-relator groups with torsion co-hopfian?*

(OR13) **(a)** *Which finitely generated one-relator groups have all generating systems (of minimal cardinality) Nielsen equivalent to each other ?*
(b) *Which finitely generated one-relator groups have only tame automorphisms (i.e., automorphisms induced by automorphisms of the ambient free group)?*

For surveys on Nielsen equivalence in groups, we refer to [166] and [74].

(OR14) (G. Baumslag, D. Spellman) *Describe one-relator groups which are discriminated by a free group.*

We note that recently, O. Kharlampovich and A. Myasnikov [100] proved that every finitely generated group which is discriminated by a free group can be obtained from a free group by applying finitely many free constructions of a very particular type.

(OR15) *If G is a one-relator group with the property that every subgroup of finite index is again a one-relator group, and every subgroup of infinite index is free, must G be a surface group?*

(OR16) *Let $S(n)$ be the orientable surface group of genus n.*

(a) *Are the groups $S(n)$ and $S(m)$ $(m, n \geq 2)$ elementary equivalent? (i.e., $Th(S(m)) = Th(S(n))$?)*
(b) *Is $S(m)$ elementary equivalent to F_{2m}, the free group of rank $2m$?*

4 Finitely Presented Groups

Although finitely generated free groups and one-relator groups are finitely presented, we believe they deserve special sections, so you won't find them here.

(FP1) *The triviality problem for groups with a balanced presentation (the number of generators equals the number of relators). See also problem (O1).*

(FP2) *Can a non-trivial finitely presented group be isomorphic to its direct square?*

We note that there are infinitely presented (but finitely generated) groups with this property – see [94]. Moreover, the same author has constructed, for any $n \geq 2$, a (infinitely presented) group G isomorphic to its nth direct power G^n, but non-isomorphic to G^k for any k, $1 < k < n$ – see [95].

(FP3) (M. Kervaire, F. Laudenbach) *Let $F_n/R = \langle x_1, ..., x_n | r_1, ..., r_m \rangle$ be a presentation of a non-trivial group. Is it true that a group $\langle x_1, ..., x_n, x_{n+1} | r_1, ..., r_m, s \rangle$ is also non-trivial for any element s from F_{n+1} ?*

It is clear from considering abelianization that, if $G = F_n/R$ is a counterexample, then G must be perfect, i.e., $G = [G, G]$. Also, it suffices to consider the case where G is an infinite simple group.

There are several related problems about (systems of) equations over groups. We only give one of them here; it appears as Problem 2a on Lyndon's list [123]:

If, in the above notation, the sum of exponents on x_{n+1} in s is not 0, does the equation $s = 1$ always have a solution over G?

For results on the latter problem, we refer to [107] and [34]. For other related problems, we refer to [82].

****(FP4)** (R. Bieri, R. Strebel)* *Is it true that if the relation module of a group G is finitely generated, then G is finitely presented ?*

M. Bestvina and N. Brady [21] gave a negative solution of this problem. Explicit presentations of their groups were later given by W. Dicks and I. Leary [44].

(FP5) (J. Stallings) *If a finitely presented group is trivial, is it always possible to replace one of the defining relators by a primitive element without changing the group?*

It is easy to see that the answer is affirmative if the group has rank 2; this is due to the fact that in the free group of rank 2, any element is in the normal closure of a primitive element.

(FP6) (C.Y.Tang) *Is there a non-free non-cyclic finitely presented group all of whose proper subgroups are free?*

(FP7) *Is every knot group virtually free-by-cyclic?*

For various properties of knot groups, we refer to the book [151].

***(FP8)** (G. Baumslag) *Is every finitely generated group discriminated by a free group, finitely presented?*

It is: see [100].

***(FP9)** (G. Baumslag) *Is a finitely generated free-by-cyclic group finitely presented?*

It is; see [53].

(FP10) (G. Baumslag, F. B. Cannonito, C. F. Miller) *Is every countable locally linear group embeddable in a finitely presented group?*

(FP11) (A.Olshanskii) *If a relatively free group is finitely presented, is it virtually nilpotent?*

(FP12) (S. Ivanov) *Is every finitely presented Noetherian group virtually polycyclic?*

(FP13) (M. I. Kargapolov) *Is every residually finite Noetherian group virtually polycyclic?*

Compare to the problem (FP12).

(FP14) (J. Wiegold) *Is every finitely generated perfect group G (i.e., $[G, G] = G$) the normal closure of a single element?*

***(FP15)** (P. Scott) *Let p, q, r be distinct prime numbers. Is the free product $Z_p * Z_q * Z_r$ the normal closure of a single element?*

No, it is not; see [83].

(FP16) (D.Anosov) *Is there a non-cyclic finitely presented group each element of which is a conjugate of some power of a single element?*

(FP17) (V. N. Remeslennikov) *Is every countable abelian group embeddable in the centre of some finitely presented group?*

***(FP18)** (R. Hirshon) *Let G be a finitely generated residually finite group, and ϕ an endomorphism of G. Is it true that $\phi^{k+1}(G)$ is isomorphic to $\phi^k(G)$ for some k?*

R. Hirshon himself [79] proved the assertion in the case where $\phi(G)$ has finite index in G. However, the answer is negative in general [182].

(FP19) (J. Makowsky) *Is there an infinite finitely presented group with finitely many conjugacy classes ?*

J. Makowsky [131] pointed out that the affirmative answer to this problem would give an example of a complete finitely axiomatizable theory T which is categorical in uncountable cardinals but not ω–categorical. Subsequently, examples of such theories were given by Peretyat'kin (1980) and others (see [Hodges, *Model theory*, p. 619]).

We also note that S. Ivanov has constructed examples, for big numbers p, of finitely generated (but infinitely presented) infinite groups of period p with precisely p conjugacy classes. These examples are included as Theorem 41.2 in [156].

(FP20) (V. Guba) *Is there a finitely generated group, other than Z_2, with exactly 2 conjugacy classes ?*

Compare to the problem (FP19).

(FP21) (E. Zelmanov) *Let F_n be the free group of rank n, and $P_{m,n}$ the subgroup of F_n generated by mth powers of all primitive elements of F_n. (This subgroup is obviously normal in F_n). Is it true that the factor group $BP(n, m) = F_n/P_{m,n}$ is NOT residually finite for sufficiently large m ?*

(FP22) *Are there finitely generated 2-relator groups with unsolvable word problem ?*

*(**FP23**) (B. Fine) *Let G be an n-generator group. Call a set of elements $\{g_1, ..., g_k\}$, $k \leq n$, a test set for the group G if, whenever $\varphi(g_i) = g_i$, $i = 1, ..., k$, for some endomorphism f of the group G, this φ is actually an automorphism of G. The test rank of G is the minimal cardinality of a test set. Can the test rank be equal to 2 if $n > 2$?*

A single element $g \in G$ is called a *test element* (see [170]) if, whenever $\phi(g) = g$ for some endomorphism φ of the group G, this φ is an automorphism of G. Thus, if G has a test element, the test rank of G is 1. For example, any free group of finite rank has test rank 1. On the other hand, there are groups (for example, free abelian groups of finite rank) whose test rank equals their rank. (Obviously, it cannot be bigger than that.)

E. I. Timoshenko [180] proved that a free metabelian group of rank > 2 has test rank 2.

C. F. Rocca Jr. and E. Turner [164] have shown recently that for any pair of integers (k, n) with $1 \leq k \leq n$, there are finitely generated abelian groups of rank n and test rank k.

(FP24) *Let G be a finitely presented group, H the intersection of all normal subgroups of finite index in G. Can G/H have unsolvable word problem?*

(FP25) (R. I. Grigorchuk) *Is it true that every finitely presented group contains either a free 2-generator semigroup, or a nilpotent subgroup of finite index?*

5 Hyperbolic and Automatic Groups

(H1) **(a)** *Are hyperbolic groups residually finite?*
(b) *Does every hyperbolic group have a proper subgroup of finite index?*

I. Kapovich and D. Wise [96] proved the equivalence of (a) and (b).

(H2) *Are hyperbolic groups linear?*

(H3) Do hyperbolic groups with torsion have solvable isomorphism problem?

We note that Z. Sela [168] has solved the isomorphism problem for torsion-free hyperbolic groups that do not split (as an amalgamated product or an HNN extension) over the trivial or the infinite cyclic group.

(H4) (A. Myasnikov) *Given a finite presentation of a hyperbolic group (which is not necessarily a Dehn presentation), is it possible to find a Dehn presentation for this group in polynomial time?*

Note that every hyperbolic group has a Dehn presentation – see [125].

(H5) (A. Myasnikov) *Given a finite presentation of an automatic group, can one decide if this group is hyperbolic?*

P. Papasoglu [157] gave a partial algorithm to recognize hyperbolic groups. Given a finite presentation $\langle S, R \rangle$, the algorithm terminates if the group $G = \langle S, R \rangle$ is hyperbolic and gives an estimate of the hyperbolicity constant δ.

(H6) (S. Gersten) *Are all automatic groups biautomatic?*

For a background on problems (H6) through (H10) we refer to [60].

(H7) (S. Gersten) *Does every automatic group have a solvable conjugacy problem?*

(H8) (S. Gersten) *Is every biautomatic group which does not contain any $Z \times Z$ subgroups, hyperbolic?*

(H9) (S. Gersten) *Can the group $< x, y; yxy^{-1} = x^2 >$ be a subgroup of an automatic group?*

(H10) (S. Gersten) *Is a retract of an automatic group automatic?*

(H11) *Does every hyperbolic group act properly discontinuously and cocompactly by isometries on a $CAT(k)$ space, where $k < 0$?*

(H12) (G. Baumslag, A. Myasnikov, V. Remeslennikov) *Is every hyperbolic group equationally noetherian? (A group is called equationally noetherian if every system of equations in finitely many variables in this group is equivalent to a finite subsystem).*

For a background on equationally noetherian groups, we refer to [17]. Here we just mention that free groups are equationally noetherian; this is due to R. Bryant [29] and V. S. Guba [71].

For a general discussion on this and related problems, we refer to [16].

(H13) *Are combable groups automatic?*

(H14) (A. Myasnikov) *We call a subgroup H of a group G malnormal if for any element g in G, but not in H, one has $H^g \cap H = \{1\}$. Is this property algorithmically decidable for finitely generated subgroups of a hyperbolic group?*

We note that malnormality is decidable in *free* groups – see [15]. Another idea on how to check malnormality is essentially contained in [174].

On the other hand, there is no algorithm that determines for *any* hyperbolic group G and its arbitrary finitely generated subgroup H whether H is malnormal in G or not – see [28].

We note that there exists an algorithm (due to D.Holt) which decides whether or not a given finitely generated *quasiconvex* subgroup of a hyperbolic group is malnormal.

(H15) (A. Myasnikov) *If a finitely generated subgroup H of a hyperbolic group is malnormal (see above), does it follow that H is quasiconvex?*

(H16) *Is every metabelian automatic group virtually abelian ?*

Note that a finitely generated nilpotent group is automatic if and only if it is virtually abelian – see [51].

6 Braid Groups

See also problem (O10).

***(B1)** *Are braid groups linear?*

There are two canonical representations of braid groups by matrices over Laurent polynomial rings – the Burau and Gassner representations (the latter is actually a representation of the pure braid group which is a subgroup of finite index in the whole braid group). Both of these representations are faithful for $n = 2, 3$ (a general reference here is [26]). A proof of the Gassner representation being faithful for every n (which implies braid groups being linear) was claimed in [6]. However, there is a controversy around this paper since several people believe they have found essential gaps in the proof (see J. S. Birman's review article 98h:20061 in Math. Reviews). This makes us consider Problem (B2) open.

Problem (B1) has been recently settled in the affirmative by S. Bigelow [25] and D. Krammer [109], who proved that the Krammer representation of the braid group B_n is faithful for every n, and therefore all braid groups are linear. For $n = 4$, see also [108].

(B2) *Is the Gassner representation of the pure braid group P_n faithful for every n ?*

See the background to (B1).

(B3) *Is the Burau representation of the braid group B_n faithful for $n = 4$?*

The Burau representation was shown to be non-faithful for $n \geq 10$ in [143], and then for $n \geq 6$ in [119]. More recently, S. Bigelow [24] has shown that the answer is negative for $n = 5$ as well.

On the other hand, it is known that the Burau representation is faithful for $n = 3$ [128].

(B4) (J. Birman) *Let $F = F_n$ be the free group of rank n generated by $a_1, ..., a_n$. Is there a solution of the equation $y_1 a_1 y_1^{-1} ... y_n a_n y_n^{-1} = a_1 ... a_n$ with all y_i from the second commutator subgroup F'' ?*

The answer is "no" if and only if the Gassner representation of the pure braid group P_n is faithful – cf. problem (B2).

(B5) (J. Birman) *Give necessary and sufficient conditions for a square matrix over Laurent polynomial ring to be the Burau matrix of some braid.*

For a background, see [26].

(B6) (V.Lin) *Let* $n \geq 4$.
(a) *Does the braid group* B_n *have a non-trivial non-injective endomorphism?*
(b) *Is it true that every non-trivial endomorphism of the commutator subgroup* $[B_n, B_n]$ *is an automorphism ?*

For a background and discussion on the problems (B6)–(B8), we refer to a recent preprint by V.Lin [116] which can be either found on the Max Planck Institut für Mathematik electronic preprint server, or requested from the author at *vlin@techunix.technion.ac.il.* Here we only note that automorphisms of braid groups were described in [50].

***(B7)** (V.Lin) *Let* $n \geq 4$.
(a) *Does the braid group* B_n *have a proper torsion-free non-abelian factor group?*
(b) *Does the commutator subgroup* $[B_n, B_n]$ *have a proper torsion-free factor group?*

S. Humphries [85] has constructed a representation of the group B_n which is shown to provide torsion-free non-abelian factor groups of B_n as well as of the commutator subgroup $[B_n, B_n]$ for $n < 7$. It is likely that the same representation should work for other values of n as well.

(B8) (V.Lin) *Let* $n \geq 4$.
(a) *Is it true that every automorphism of the commutator subgroup* $[B_n, B_n]$ *can be extended to an automorphism of the whole group* B_n *?*
(b) *Is it true that every non-trivial endomorphism of* $[B_n, B_n]$ *can be extended to an endomorphism of* B_n *?*

(B9) (P. Dehornoy) *We call a braid word* w σ*-positive (resp.* σ*-negative) if the generator* σ_i *with minimal index occurs only with positive (resp. negative) exponents in* w*. Is it true that, for every* n*, there exists a constant* $c(n)$ *such that every* n*-strand braid word of length* N *is equivalent to a* σ*-positive or a* σ*-negative braid word of length* $c(n) \cdot N$ *at most ?*

(B10) (P. Dehornoy) *Say that a braid is special if it can be obtained from the trivial braid by using iteratively the self-distributive exponentiation* $a \wedge b = a\ \Omega(b)\ \sigma_1\ \Omega(a)^{-1}$*, where* Ω *is the shift endomorphism of* B_∞ *that maps* σ_i *to* σ_{i+1} *for every* i*. How many special braids are there in* B_n *?*

For a general background to the problems (B9) and (B10), we refer to the monograph [39].

(B11) (G. Makanin) *Is it true that in any group* B_n, $g^k = h^k$ *for some* $k \neq 0$ *implies that* g *is conjugate to* h *?*

(B12) (V. Shpilrain) *Let* K_n *be the kernel of the Burau representation of the braid group* B_n, $n > 4$. *Is the factor group* B_n/K_n *torsion free ?*

An affirmative answer to this problem would also imply a solution of the problem (B7).

7 Nilpotent Groups

***(N1)** (A. Myasnikov) *Let* G *be a free nilpotent group of finite rank. Suppose an element* $g \in G$ *is fixed by every automorphism of* G. *Is it true that* $g = 1$*?*

V.Bludov has communicated the following example of a non-trivial element g of a free nilpotent group of rank 2 and nilpotency class $k \geq 8$, which is fixed by every automorphism: $g = [a, [a, b], [a, b, b], [a, b], ..., [a, b]]$, where there are $(2k-3)$ occurrences of $[a, b]$ after $[a, b, b]$. (Here a and b are generators of the free nilpotent group).

Recently, A. Papistas [158] and, independently, E. Formanek [54] have solved this problem completely by classifying all pairs (r, c) for which $F(r, c)$, the free nilpotent group of rank r and class c, has nontrivial elements fixed by all automorphisms.

(N2) *Let* G *be a finitely generated nilpotent group. Is the Dehn function of* G *equivalent to a polynomial?*

Let $P = \langle X \mid R \rangle$ be a finite presentation of a group G, $F(X)$ a free group on X, and $ncl(R)$ the normal closure of R in $F(X)$. The "area" of $w \in ncl(R)$ is defined by

$$A(w) = min\{ \ m \ \mid \ w = \prod_{i=1}^{m} c_i^{-1} r_i^{e_i} c_i, \ c_i \in F(X), r_i \in R, e_i = \pm 1\}.$$

Now, the *isoperimetric function* of the presentation P is given by

$$\Phi_P(n) = max\{ \ A(w) \ \mid \ w \in ncl(R), \ |w| \leq n\},$$

where $|w|$ is the length of w in $F(X)$.

Let N be the set of all non-negative integers. For functions $f, h : N \to N$ we define a relation $f \preceq h$ iff there exists a constant K such that $f(n) \leq K \cdot h(Kn) + Kn$ for every $n \in N$. We write $f \simeq h$ iff $f \preceq h$ and $h \preceq f$. It is not hard to show that if P and Q are two finite presentations of a group G, then $\Phi_P \simeq \Phi_Q$. Any function equivalent to Φ_P is called the *Dehn function* of G. From now on, we shall denote the Dehn function of a group G by Φ_G.

S. Gersten [59] proved that for any finitely generated nilpotent group G, Φ_G is bounded by a polynomial of degree 2^h, where h is the Hirsch length of G. G. Conner [36] improved the bound on the degree to 2^c, where c is the nilpotency class of G. Recently, C.Hidber [78] proved that $\Phi_G \preceq n^{2c}$. It is known that if G is a free nilpotent group of class c, then $\Phi_G \simeq n^{c+1}$, in particular, Φ_G is equivalent to a polynomial. Whether or not this is true in general, is still an open problem.

(N3) (B. I. Plotkin) *Is it true that every locally nilpotent group is a homomorphic image of a torsion-free locally nilpotent group?*

(N4) (G. Baumslag) *Let G be a finitely generated torsion-free nilpotent group. Is it true that there are only finitely many non-isomorphic groups in the sequence $Aut(G)$, $Aut(Aut(G))$, ... ?*

J. Hamkins [76] established the property in the title of his paper.

(N5) (G. Baumslag) *Is the property of being directly indecomposable decidable for finitely generated nilpotent groups?*

(N6) (A. Myasnikov) *Describe all finitely generated nilpotent groups of class 2 which have genus 1. (We say that a group G has genus 1 if every group with the same set of finite homomorphic images as G, is isomorphic to G).*

See the background to the problem (F14).

(N7) *Is every group with an Engel identity $[x, y, ..., y] = 1$, locally nilpotent?*

8 Metabelian Groups

Some of the problems about free groups (particularly (F1), (F3)) are also of interest when asked about free metabelian groups.

(M1) *The isomorphism problem for finitely presented metabelian groups.*

There is an algorithm to determine whether or not a given finitely generated metabelian group is free metabelian – see [70] and the paper by Noskov cited in the background to the problem (F14).

We also note that "most" algorithmic problems about finitely presented metabelian groups are solvable – see [13] and references thereto.

(M2) *Is the automorphism group of a free metabelian group of rank > 3 finitely presented ?*

The automorphism group of a free metabelian group of finite rank is known to be finitely generated unless the rank equals 3 – see [9] and [8].

(M3) (F. B. Cannonito) *Is there an algorithm which decides whether or not a given finitely presented solvable group is metabelian?*

(M4) (P. Hall) *Are projective groups of infinite countable rank in the class of metabelian groups free metabelian?*

For groups of finite rank, the answer is affirmative – see [4].

(M5) (G. Baumslag) *What can one say about the integral homology of a finitely generated metabelian group?*

For a survey on homological properties of metabelian groups, we refer to [111].

***(M6)** (V. Shpilrain) Is it true that every IA-automorphism of a free metabelian group of finite rank has a non-trivial fixed point?

For a background, we refer to [173].
The problem was recently answered in the negative by M. Kassabov [97].

***(M7)** (R. Goebel) *Is there a group which is NOT isomorphic to the outer automorphism group of any metabelian group with a trivial centre?*

No, there is no such group – see [62].

9 Solvable Groups

(S1) (A. I. Mal'cev) *Describe the automorphism group of a free solvable group of finite rank. In particular, is this group finitely generated?*

The automorphism group of a free solvable group of derived length > 2 and *rank* > 2 cannot be generated by elementary Nielsen automorphisms – see [72] and [169]. Moreover, every free solvable group of derived length $d > 2$ and rank $r > 2$ has automorphisms that cannot be lifted to automorphisms of the free solvable group of derived length $d + 1$ and the same rank r – see [171]. It is not known however whether or not those automorphism groups are finitely generated.

(S2) (M. I. Kargapolov) *The word problem for groups admitting a single defining relation in the variety of all solvable groups of a given derived length.*

We note that the word problem for groups admitting *finitely many* defining relations in the variety of all solvable groups of a given derived length > 2 is, in general, unsolvable – see [99].

(S3) (M. I. Kargapolov) *Is it true that every group of rank > 2 admitting a single defining relation in the variety of all solvable groups of a given derived length, has trivial centre?*

E.Timoshenko [179] settled this problem in the affirmative for metabelian groups. C. K. Gupta and V. Shpilrain [73] settled the problem (also in the affirmative) for solvable groups of arbitrary derived length, under an additional assumption that the relator is not a proper power modulo any term of the derived series.

(S4) (M. I. Kargapolov) *Is there a number $N = N(k, d)$ such that every element of the commutator subgroup of a free solvable group of rank k and derived length d, is a product of N commutators?*

The answer is "yes" for free metabelian groups – see [2] and for free solvable groups of derived length 3 – see [163].

(S5) (P. M. Neumann) *Is it true that if A, B are finitely generated solvable Hopfian groups, then $A \times B$ is Hopfian?*

(S6) (V. N. Remeslennikov) *The conjugacy problem for finitely generated abelian-by-polycyclic groups.*

(S7) (D. Robinson) *Is there a finitely presented solvable group satisfying the maximum condition on normal subgroups, with unsolvable word problem?*

(S8) (G. Baumslag, V. Remeslennikov) *Is a finitely generated free solvable group of derived length 3 embeddable in a finitely presented solvable group?*

***(S9)** (B. Fine, V. Shpilrain) *Let u be an element of a group G. We call u a test element if, whenever $\phi(u) = u$ for some endomorphism ϕ of the group G, this ϕ is actually an automorphism of G. Does the free solvable group of rank 2 and derived length $d > 2$ have any test elements?*

The most obvious candidate for a test element in a group generated by x and y would be $u = [x, y]$. This however is *not* a test element in a free solvable group of derived length $d > 2$ – see [75]

Recently, V. Roman'kov [165] has constructed test elements in the free solvable group of rank 2 and derived length 3.

It is plausible that the same method can be used for constructing test elements in the free solvable group of any bigger rank as well, but technically it is getting more complicated.

We also mention a related result of E. I. Timoshenko [180] who proved that a free metabelian group of rank > 2 does *not* have any test elements. It was previously known [47] that the free metabelian group of rank 2 does have test elements, for example, $u = [x, y]$.

10 Groups of Matrices

(MA1) *Is the group $GL_2(Z[t, t^-1])$, i.e., the group of invertible matrices over the ring of one-variable Laurent polynomials with integral coefficients, generated by elementary and diagonal matrices ?*

***(MA2)** *Find a particular matrix from $GL_2(Z[t, t^-1, s, s^-1])$, which is not a product of elementary and diagonal matrices.*

M. Evans [52] has recently found such matrices. It was previously known due to [7] that such matrices do exist.

(MA3) *The generalized word problem for the group $SL_3(Z)$.*

We note that for $SL_4(Z)$, the generalized word problem is unsolvable since a direct product of two free groups embeds into $SL_4(Z)$.

(MA4) (S. Thomas) *Does there exist a simple torsion-free linear group ?*

(MA5) (A. L. Shmelkin) *Is it true that identities of any linear group have a finite basis ?*

11 Growth

(G1) (S. I. Adian) *Is it true that a finitely presented group has either polynomial or exponential growth ?*

The point here is that there are examples of groups of intermediate growth (between polynomial and exponential), but all these groups are infinitely presented – see [65], [66], [67], [69].

(G2) (R. I. Grigorchuk) **(a)** *Is it true that there is a gap between polynomial rate of growth and the rate of growth of the function $e^{\sqrt{n}}$ on the scale of growth of finitely generated groups?*
(b) *Is there a finitely generated group G whose growth function is equivalent to $e^{\sqrt{n}}$?*

(G3) (M. Gromov, J. Lafontaine, P. Pansu) *Is there a group of exponential, but not uniformly exponential growth ?*

A group G has uniformly exponential growth if there is $c > 1$ such that, for any generating system of G, the growth function of G with respect to this generating system is at least $O(c^n)$.

(G4) (R. I. Grigorchuk) *Is it true that every finitely generated infinite simple group has exponential growth ?*

(G5) (R. I. Grigorchuk) *Is it true that every hereditary just infinite group (i.e., a residually finite group whose every subgroup of finite index is just infinite) has either polynomial or exponential growth ?*

(G6) (R. I. Grigorchuk) *Is there a cancellative semigroup of subexponential growth whose quotient group (if it exists) has exponential growth ?*

(G7) (R. I. Grigorchuk) *Let G be a group of subexponential growth, and let $f(n)$ be the number of different elements of length at most n in the group G.*

Does the limit (when $n \to \infty$) of the ratio $f(n+1)/f(n)$ always exist ?

References

[1] S. Akbulut, R. Kirby, *A potential smooth counterexample in dimension 4 to the Poincare conjecture, the Schoenflies conjecture, and the Andrews-Curtis conjecture.* Topology **24** (1985), 375–390.

[2] Kh. S. Allambergenov, V. A. Romankov, *Products of commutators in groups* (Russian). Dokl. Akad. Nauk UzSSR 1984, 14–15.

[3] J. J. Andrews and M. L. Curtis, *Free groups and handlebodies*, Proc. Amer. Math. Soc. **16** (1965), 192–195.

[4] V. A. Artamonov, *Projective metabelian groups and Lie algebras* (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 226–236, 469.

[5] M. Auslander, R. C. Lyndon, *Commutator subgroups of free groups*, Amer. J. Math. **77** (1955), 929–931.

[6] S. Bachmuth, *Braid groups are linear groups*, Adv. Math. **121** (1996), 50–61.

[7] S. Bachmuth, H. Mochizuki, $E_2 \neq \mathrm{SL}_2$ *for most Laurent polynomial rings*, Amer. J. Math. **104** (1982), 1181–1189.

[8] S. Bachmuth, H. Mochizuki, *The nonfinite generation of* $\mathrm{Aut}(G)$*, G free metabelian of rank* 3, Trans. Amer. Math. Soc. **270** (1982), 693–700.

[9] S. Bachmuth, H. Mochizuki, $\mathrm{Aut}(F) \to \mathrm{Aut}(F/F")$ *is surjective for free group F of rank* ≥ 4, Trans. Amer. Math. Soc. **292** (1985), 81–101.

[10] G. Baumslag, *Some aspects of groups with unique roots*, Acta Math. **104** (1960), 217–303.

[11] G. Baumslag, *On the residual nilpotence of certain one-relator groups*, Comm. Pure Appl. Math. **21** (1968), 491–506.

[12] G. Baumslag, *Some open problems.* Summer School in Group Theory in Banff, 1996, 1–9. CRM Proceedings and Lecture Notes. **17**. Amer. Math. Soc., Providence, 1999.

[13] G. Baumslag, F. Cannonito, D. Robinson, *The algorithmic theory of finitely generated metabelian groups*, Trans. Amer. Math. Soc. **344** (1994), 629–648.

[14] G. Baumslag, S. M. Gersten, M. Shapiro, H. Short, *Automatic groups and amalgams – a survey*, Algortims and Classification in Combinatorial Group Theory (Berkeley, CA, 1989), 179–194, Math. Sci. Res. Inst. Publ., **23.** Springer, New York, 1992.

[15] G. Baumslag, A. Myasnikov, V. Remeslennikov, *Malnormality is decidable in free groups*, Internat. J. Algebra Comput. **9** (1999), 687–692.

[16] G. Baumslag, A. Myasnikov, V. Remeslennikov, *Algebraic geometry over groups I: Algebraic sets and ideal theory*, J. Algebra **219** (1999), 16–79.

[17] G. Baumslag, A. Myasnikov, V. Roman'kov, *Two theorems about equationally Noetherian groups*, J. Algebra **194** (1997), 654–664.

[18] G. Baumslag, A. G. Myasnikov, V. Shpilrain, *Open problems in combinatorial group theory*, in: Groups, languages and geometry (South Hadley, MA, 1998), 1–27. Contemp. Math. **250**, Amer. Math. Soc., 1999.

[19] G. Baumslag, D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201.

[20] G. Bergman, *Supports of derivations, free factorizations, and ranks of fixed subgroups in free groups*, Trans. Amer. Math. Soc. **351** (1999), 1531–1550.

[21] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), 445–470.

[22] M. Bestvina, M. Feighn, M. Handel, *The Tits Alternative for Out(F_n). I : Dynamics of exponentially growing automorphisms*, Ann. of Math. **151** (2000), 517–623.

[23] M. Bestvina and M. Handel, *Train tracks and automorphisms of free groups*, Ann. of Math. **135** (1992), 1–53.

[24] S. Bigelow, *The Burau representation is not faithful for $n = 5$*, Geom. Topol. **3** (1999), 397–404 (electronic).

[25] S. Bigelow, *Braid groups are linear*, J. Amer. Math. Soc. **14** (2001), 471–486.

[26] J. S. Birman, *Braids, links and mapping class groups*, Ann. Math. Studies **82**, Princeton Univ. Press, 1974.

[27] A. Borovik, A. G. Myasnikov, V. Shpilrain, *Measuring sets in infinite groups*, Contemp. Math., to appear.

[28] M. Bridson, D. Wise, *Malnormality is undecidable in hyperbolic groups*, Israel J. Math. **124** (2001), 313–316.

[29] R. Bryant, *The verbal topology of a group.* J.Algebra **48** (1977), 340–346.

[30] M. Burger, S. Mozes, *Finitely presented simple groups and products of trees*, C. R. Acad. Sci. Paris Sr. I Math. **324** (1997), 747–752.

[31] R. Burns, *On the intersection of finitely generated subgroups of a free group*, Math. Z. **119** (1971), 121–130.

[32] R. G. Burns, O. Macedonska, *Balanced presentations of the trivial group*, Bull. London Math. Soc. **25** (1993), 513–526.

[33] R. Camm, *Simple free products.* J. London Math. Soc. **28** (1953), 66–76.

[34] A. Clifford, R. Z. Goldstein, *Tesselations of S^2 and equations over torsion-free groups*, Proc. Edinburgh Math. Soc. (2) **38** (1995), 485–493.

[35] M. Cohen and M. Lustig, *On the dynamics and the fixed subgroup of a free group automorphism.* Invent. Math. **96** (1989), 613–638.

[36] G. Conner, *Central extensions of word hyperbolic groups satisfy a quadratic isoperimetric inequality*, Arch. Math. **65** (1995), 465–479.

[37] D. Cooper, *Automorphisms of free groups have finitely generated fixed point sets*, J. Algebra **111** (1987), 453–456.

[38] T. Coulbois, A. Khelif, *Equations in free groups are not finitely approximable*, Proc. Amer. Math. Soc. **127** (1999), 963–965.

[39] P. Dehornoy, *Braids and self-distributivity.* Progress in Mathematics, **192**. Birkhäuser Verlag, Basel, 2000.

[40] T. Delzant, *Group rings of hyperbolic groups*, C. R. Acad. Sci. Paris Sr. I Math. **324** (1997), 381–384.

[41] W. Dicks, *Equivalence of the strengthened Hanna Neumann conjecture and the amalgamated graph conjecture,* Invent. Math. **117** (1994), 373–389.

[42] W. Dicks, *Erratum: W. S. Jassim*, Rev. Mat. Univ. Complut. Madrid **11** (1998), 263–265.

[43] W. Dicks and E. Formanek, *The rank three case of the Hanna Neumann conjecture,* J. Group Theory **4** (2001), 113–151.

[44] W. Dicks and I. Leary, *Presentations for subgroups of Artin groups*, Proc. Amer. Math. Soc. **127** (1999), 343–348.

[45] W. Dicks, E. Ventura, *The group fixed by a family of injective endomorphisms of a free group*, Contemporary Math. **195** (1996).

[46] M. Dunwoody, *On verbal subgroups of free groups*, Arch. Math. **16** (1965), 153–157.

[47] V. G. Durnev, *The Mal'tsev-Nielsen equation on a free metabelian group of rank 2*, Math. Notes USSR **46** (1989), 927–929.

[48] V. G. Durnev, *A generalization of Problem 9.25 in the Kourovka notebook*, Math. Notes USSR **47** (1990), 117–121.

[49] J. Dyer, E. Formanek and E. Grossman, *On the linearity of automorphism groups of free groups*, Arch. Math. **38** (1982), 404–409.

[50] J. Dyer, E. Grossman, *The automorphism groups of the braid groups,* Amer. J. Math. **103** (1981), 1151–1169.

[51] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, *Word processing in groups.* Jones and Bartlett Publishers, Boston, 1992.

[52] M. Evans, *Primitive elements in the free metabelian group of rank 3*, J.Algebra **220** (1999), 475–491.

[53] M. Feighn, M. Handel, *Mapping tori of free group automorphisms are coherent*, Ann. of Math. **149** (1999), 1061–1077.

[54] E. Formanek, *Fixed points and centers of automorphism groups of free nilpotent groups*, Comm. Algebra, to appear.

[55] E. Formanek and C. Procesi, *The automorphism group of a free group is not linear,* J. Algebra **149** (1992), 494–499.

[56] A. M. Gaglione, A. G. Myasnikov, V. N. Remeslennikov, D. Spellman, *Formal power series representations of free exponential groups*, Comm. Algebra **25** (1997), 631–648.

[57] S. Gersten, *On fixed points of automorphisms of finitely generated free groups*, Bull. Amer. Math. Soc. **8** (1983), 451–454.

[58] S. Gersten, *Fixed points of automorphisms of free groups*, Adv. in Math. **64** (1987), 51–85.

[59] S. Gersten, *Isodiametric and isoperimetric inequalities in group extensions.* Preprint, University of Utah, 1991.

[60] S. M. Gersten, *Problems on automatic groups.* Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), 225–232, Math. Sci. Res. Inst. Publ. **23**, Springer, New York, 1992.

[61] D. Gildenhuys, O. Kharlampovich and A. Myasnikov, *CSA groups and separated free constructions*, Bull. Austr. Math. Soc. **52** (1995), 63–84.

[62] R. Goebel, A. Paras, *Outer automorphism groups of metabelian groups*, J. Pure Appl. Algebra **149** (2000), 251–266.

[63] R. Goldstein and E. Turner, *Fixed subgroups of homomorphisms of free groups*, Bull. London Math. Soc. **18** (1986), 468–470.

[64] E. Golod, *On nil-algebras and finitely approximable p-groups* (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 273–276.

[65] R. I. Grigorchuk, *On the Milnor problem of group growth* (Russian), Dokl. Akad. Nauk SSSR **271** (1983), 30–33.

[66] R. I. Grigorchuk, *Construction of p-groups of intermediate growth that have a continuum of factor-groups* (Russian), Algebra i Logika **23** (1984), 383–394, 478.

[67] R. I. Grigorchuk, *Degrees of growth of p-groups and torsion-free groups* (Russian), Mat.Sb. **126(168)** (1985), 194–214, 286.

[68] R. I. Grigorchuk, P. F. Kurchanov, *Some questions of group theory related to geometry.* Translated from the Russian by P.M.Cohn. Encyclopaedia Math. Sci., **58**, Algebra, VII, 167–240, Springer, Berlin, 1993.

[69] R. I. Grigorchuk, A. Maki, *On a group of intermediate growth that acts on a line by homeomorphisms* (Russian), Mat. Zametki **53** (1993), 46–63; translation in Math. Notes **53** (1993), 146–157.

[70] J. R. J. Groves, C. F. Miller III, *Recognizing free metabelian groups*, Illinois J. Math. **30** (1986), 246–254.

[71] V. S. Guba, *Equivalence of infinite systems of equations in free groups and semigroups to finite subsystems*, Math. Notes USSR **40** (1986), 688–690.

[72] C. K. Gupta, F. Levin, *Tame range of automorphism groups of free polynilpotent groups*, Comm. Algebra **19** (1991), 2497–2500.

[73] C. K. Gupta and V. Shpilrain, *The centre of a one-relator solvable group*, Internat. J. Algebra Comput. **3** (1993), 51–55.

[74] C. K. Gupta, V. Shpilrain, *Lifting automorphisms: a survey*, Groups '93 Galway/St. Andrews, Vol. 1 (Galway, 1993), 249–263, London Math. Soc. Lecture Note Ser., **211**, Cambridge Univ. Press, Cambridge, 1995.

[75] N. Gupta, V. Shpilrain, *Nielsen's commutator test for two-generator groups*, Math. Proc. Cambridge Philos. Soc. **114** (1993), 295–301.

[76] J. Hamkins, *Every group has a terminating transfinite automorphism tower*, Proc. Amer. Math. Soc. **126** (1998), 3223–3226.

[77] J. Harlander, *Solvable groups with cyclic relation module*, J. Pure Appl. Algebra **90** (1993), 189–198.

[78] C. Hidber, *Isoperimetric functions of finitely generated nilpotent groups and their amalgams*, Ph.D. thesis.

[79] R. Hirshon, *Some properties of endomorphisms in residually finite groups*, J. Austral. Math. Soc. Ser. A **24** (1977), 117–120.

[80] C. Hog-Angeloni, W. Metzler, *The Andrews-Curtis conjecture and its generalizations*, Two-dimensional homotopy and combinatorial group theory, 365–380, London Math. Soc. Lecture Note Ser., **197**, Cambridge Univ. Press, Cambridge, 1993.

[81] J. Howie, *Some remarks on a problem of J. H. C. Whitehead,* Topology **22** (1983), 475–485.

[82] J. Howie, *How to generalize one-relator group theory.* Combinatorial group theory and topology (Alta, Utah, 1984), 53–78, Ann. of Math. Stud., **111**, Princeton Univ. Press, 1987.

[83] J. Howie, *A proof of the Scott-Wiegold conjecture on free products of cyclic groups*, J. Pure Appl. Algebra, to appear.

[84] J. Huebschmann, *Aspherical* 2*-complexes and an unsettled problem of J. H. C. Whitehead,* Math. Ann. **258** (1981/82), 17–37.

[85] S. Humphries, *Torsion-free quotients of braid groups*, Internat. J. Algebra Comput. **11** (2001), 363–373.

[86] W. Imrich and E. C. Turner, *Endomorphisms of free groups and their fixed points*, Math. Proc. Cambridge Phil. Soc. **105** (1989), 421–422.

[87] S. V. Ivanov, *The free Burnside groups of sufficiently large exponents.* Internat. J. Algebra Comput. **4** (1994), 308 pp.

[88] S. Ivanov, *On certain elements of free groups*, J. Algebra **204** (1998), 394–405.

[89] S. Ivanov, *On endomorphisms of a free group that preserve primitivity*, Arch. Math. **72** (1999), 92–100.

[90] S. Ivanov, *An asphericity conjecture and Kaplansky problem on zero divisors*, J. Algebra **216** (1999), 13–19.

[91] S. V. Ivanov, P. E. Schupp, *On the hyperbolicity of small cancellation groups and one-relator groups*, Trans. Amer. Math. Soc. **350** (1998), 1851–1894.

[92] S. V. Ivanov and P. Schupp, *A remark on finitely generated subgroups of free groups*, in: Algorithmic Problems in Groups and Semigroups, Birkhauser, 2000, pp. 139–142.

[93] W. S. Jassim, *On the intersection of finitely generated subgroups of free groups*, Rev. Mat. Univ. Complut. Madrid **9** (1996), 67–84.

[94] J. M. T. Jones, *Direct products and the Hopf property*, J. Austral. Math. Soc. **17** (1974), 174–196.

[95] J. M. T. Jones, *On isomorphisms of direct powers.* Word problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976), 215–245, Stud. Logic Foundations Math., **95**, North-Holland, 1980.

[96] I. Kapovich and D. Wise, *The equivalence of some residual properties of word-hyperbolic groups*, J. Algebra **223** (2000), 562–583.

[97] M. Kassabov, An automorphism of a free metabelian group without fixed points, preprint.

[98] B. Khan, *Positively generated subgroups of free groups and the Hanna Neumann conjecture,* this volume.

[99] O. G. Kharlampovich, *A finitely presented solvable group with unsolvable word problem* (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), 852–873.

[100] O. Kharlampovich, A. Myasnikov, *Irreducible affine varieties over groups. I, II.* J. Algebra **200** (1998), 472–516, 517–570.

[101] O. Kharlampovich and A. Myasnikov, *Hyperbolic groups and free constructions*, Trans. Amer. Math. Soc. **350** (1998), 571–613.

[102] O. Kharlampovich and A. Myasnikov, *Tarski's problem about the elementary theory of free groups has a positive solution*, Electron. Res. Announc. Amer. Math. Soc. **4** (1998), 101–108 (electronic).

[103] O. Kharlampovich and A. Myasnikov, *Implicit function theorem over free groups and genus problem*, in: Knots, braids and mapping class groups. AMS/IP Studies in Advanced Math. **24** (2002).

[104] O. Kharlampovich and A. Myasnikov, *Equations over fully residually free groups*, preprint.

[105] O. Kharlampovich and A. Myasnikov, *Elementary theory of free nonabelian groups*, preprint.

[106] D. G. Khramtsov, *Finite groups of automorphisms of free groups*, Math. Notes **38** (1985), 721–724.

[107] A. A. Klyachko, *A funny property of sphere and equations over groups.* Comm. Algebra **21** (1993), 2555–2575.

[108] D. Krammer, *The braid group B_4 is linear*, Invent. Math. **142** (2000), 451–486.

[109] D. Krammer, *Braid groups are linear*, Ann. of Math., to appear.

[110] P. Kropholler, P. Linnell and J. Moody, *Applications of a new K-theoretic theorem to soluble group rings*, Proc. Amer. Math. Soc. **104** (1988), 675–684.

[111] Yu. V. Kuz'min, *Homology theory of free abelianized extensions*, Comm. Algebra **16** (1988), 2447–2533.

[112] I. Leary, *Asphericity and zero divisors in group algebras*, J. Algebra **227** (2000), 362–364.

[113] D. Lee, *On certain C-test words for free groups*, J. Algebra **247** (2002), 509–540.

[114] D. Lee, *Primitivity preserving endomorphisms of free groups*, Comm. Algebra, to appear.

[115] G. Levitt, M. Lustig, *Periodic ends, growth rates, Hölder dynamics for automorphisms of free groups*, Comm. Math. Helv. **75** (2000), 415–429.

[116] V. Lin, *Braids, permutations, polynomials. I*, preprint.

[117] S.Lipschutz, *Generalization of Dehn's result on the conjugacy problem*, Proc. Amer. Math.Soc. **17** (1966), 759–762.

[118] S. Lipschutz, *The conjugacy problem and cyclic amalgamations*, Bull. Amer. Math.Soc. **81** (1975), 114–116.

[119] D. Long and M. Paton, *The Burau representation is not faithful for $n \geq 6$*, Topology **32** (1993), 439–447.

[120] J. Los, *On the conjugacy problem for automorphisms of free groups*, Topology **35** (1996), 779–808.

[121] E. Luft, *On 2-dimensional aspherical complexes and a problem of J. H. C. Whitehead,* Math. Proc. Cambridge Philos. Soc. **119** (1996), 493–495.

[122] R. Lyndon, *Groups with parametric exponents*, Trans. Amer. Math. Soc. **96** (1960), 518–533.

[123] R. Lyndon, *Problems in combinatorial group theory.* Combinatorial group theory and topology (Alta, Utah, 1984), 3–33, Ann. of Math. Stud., **111**, Princeton Univ. Press, 1987.

[124] R. Lyndon, P. Schupp, *Combinatorial Group Theory*, (Reprint of the 1977 edition). In *Classics in Mathematics*, Springer-Verlag, Berlin (2001).

[125] I. G. Lysenok, *Some algorithmic properties of hyperbolic groups*, Math. USSR Izv. **35** (1990), 145–163.

[126] I. G. Lysenok, *Infinite Burnside groups of even period* (Russian). Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996), no. 3, 3–224.

[127] W. Magnus, *Über discontinuierliche gruppen mit einer definierenden Relation (Der Freiheitssatz)*, J.Reine Angew. Math. **163** (1930), 141–165.

[128] W. Magnus, A. Peluso, *On a theorem of V. I. Arnold*, Comm. Pure Appl. Math. **22** (1969), 683—692.

[129] G. Makanin, *Equations in a free group*, Math. USSR Izv. **21** (1983), no. 3, 546–582.

[130] G. Makanin, *Decidability of the universal and positive theories of a free group*, Math. USSR Izv. **25** (1985), no. 1, 75–88.

[131] J. Makowsky, *On some conjectures connected with complete sentences*, Fund. Math. **81** (1974), 193–202.

[132] A. Martino, E. Ventura, *On automorphism-fixed subgroups of a free group*, J. Algebra **230** (2000), 596–607.

[133] J. McCammond, *The Burnside groups and small cancellation theory*. Groups St. Andrews 1997 in Bath, II, 538–559, London Math. Soc. Lecture Note Ser. **261**, Cambridge Univ. Press, Cambridge, 1999.

[134] J. McCool, *A characterization of periodic automorphisms of a free group*, Trans. Amer. Math. Soc. **260** (1980), 309–318.

[135] J. McCool, *Free group roots of* $a^k b^l$ *and* $[a^k, b]$, Internat. J. Algebra Comput. **10** (2000), 339–347.

[136] J.McCool, *On a question of Remeslennikov*, preprint.

[137] J. McCool and S. Krstic, *The non-finite presentability of* $IA(F_3)$ *and* $GL_2(Z[t, t^{-1}])$, Invent. Math. **129** (1997), 595–606.

[138] J. McCool, A. Pietrowski, *On free products with amalgamation of two infinite cyclic groups*, J. Algebra **18** (1971), 377–383.

[139] J. Meakin and P. Weil, *Subgroups of free groups: a contribution to the Hanna Neumann conjecture*, Geom. Dedicata, to appear.

[140] Yu. Merzlyakov, *Positive formulae on free groups* (Russian). Algebra i Logika **4** (1966), 25–42.

[141] C. F. Miller III, *On group-theoretic decision problems and their classification*, Annals of Math. Studies **111**, Princeton Univ. Press, 1987, p.31.

[142] C. F. Miller and P. Schupp, *Some presentations of the trivial group*, Contemp. Math. **250**, Amer. Math. Soc., Providence, RI, 1999, 113–115.

[143] J. Moody, *The faithfulness question for the Burau representation*, Proc. Amer. Math. Soc. **119** (1993), 671–679.

[144] A. G. Myasnikov, *Extended Nielsen transformations and the trivial group* (Russian). Mat. Zametki **35** (1984), 491–495.

[145] A. D. Myasnikov and A. G. Myasnikov, *Balanced presentations of the trivial group on two generators and the Andrews-Curtis conjecture*, in: Groups and computation, III (Columbus, OH, 1999), 257–263, Ohio State Univ. Math. Res. Inst. Publ., **8**, de Gruyter, Berlin, 2001.

[146] A. D. Myasnikov, A. G. Myasnikov and V. Shpilrain, *On the Andrews-Curtis equivalence*, this volume.

[147] A. Myasnikov and V. Remeslennikov, *Exponential groups.II. Extensions of centralizers and tensor completion of CSA-groups*, Internat. J. Algebra Comput. **6** (1996), 687–711.

[148] A. Myasnikov and V. Shpilrain, *Automorphic orbits in free groups*, preprint.

[149] V. Nekrashevych, S. Sidki, *Automorphisms of the binary tree: state-closed subgroups and dynamics of 1/2-endomorphisms*, preprint.

[150] W. Neumann, *On intersections of finitely generated subgroups of free groups*, in: Groups–Canberra 1989, 161–170, Lecture Notes Math. **1456**, Springer, Berlin, 1990.

[151] L. P. Neuwirth, *Knot groups.* Annals of Mathematics Studies **56**, Princeton University Press, Princeton, N.J. 1965.

[152] B. B. Newman, *Some results on one-relator groups*, Bull. Amer. Math. Soc. **74** (1968), 568–571.

[153] G. A. Noskov, *The genus of a free metabelian group* (Russian). Preprint 84-509. Akad. Nauk SSSR Sibirsk. Otdel., Vychisl. Tsentr, Novosibirsk, 1984. 18 pp.

[154] P. S. Novikov and S. I. Adian, *Infinite periodic groups.* I, II, III (Russian). Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 212–244, 251–524, 709–731.

[155] A. Olshanskii, *SQ-universality of hyperbolic groups*, Mat. Sb. **186** (1995), no. 8, 119–132.

[156] A. Yu. Ol'shanskii, *Geometry of defining relations in groups.* Mathematics and its Applications (Soviet Series), **70**. Kluwer Academic Publishers Group, Dordrecht, 1991.

[157] P. Papasoglu, *An algorithm detecting hyperbolicity.* Geometric and computational perspectives on infinite groups, 193–200, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., **25**, Amer. Math. Soc., Providence, RI, 1996.

[158] A. I. Papistas, *A note on fixed points of certain relatively free nilpotent groups*, Comm. Algebra **29** (2001), 4693–4699.

[159] D. S. Passman, *The algebraic structure of group rings*, John Wiley and Sons, New York, 1977.

[160] A. Pietrowski, *The isomorphism problem for one-relator groups with non-trivial centre*, Math. Z. **136** (1974), 95–106.

[161] S. Pride, *The isomorphism problem for two-generator one-relator groups with torsion is solvable*, Trans. Amer. Math. Soc. **227** (1977), 109–139.

[162] A. Razborov, *Systems of equations in a free group*, Math. USSR Izv. **25** (1985), no. 1, 115–162.

[163] A. H. Rhemtulla, *Commutators of certain finitely generated soluble groups*, Canad. J. Math. **21** (1969), 1160–1164.

[164] C. F. Rocca Jr. and E. C. Turner, *Test ranks of finitely generated abelian groups*, this volume.

[165] V. Roman'kov, *On test elements in free solvable groups of rank 2*, Algebra i Logika **40** (2001), 192–201 (Russian).

[166] G. Rosenberger, *Minimal generating systems for plane discontinuous groups and an equation in free groups.* Groups–Korea 1988 (Pusan, 1988), 170–186, Lecture Notes in Math., **1398**, Springer, Berlin, 1989.

[167] G. Sacerdote, *Elementary properties of free groups*, Trans. Amer. Math. Soc. **178** (1973), 127–138.

[168] Z. Sela, *The isomorphism problem for hyperbolic groups*, Ann. of Math. (2) **141** (1995), 217–283.

[169] V. Shpilrain, *Automorphisms of F/R' groups*, Internat. J. Algebra Comput. **1** (1991), 177–184.

[170] V. Shpilrain, *Recognizing automorphisms of the free groups*, Arch. Math. **62** (1994), 385–392.

[171] V. Shpilrain, *Non-commutative determinants and automorphisms of groups*, Comm. Algebra **25** (1997), 559–574.

[172] V. Shpilrain, *Generalized primitive elements of a free group*, Arch. Math. **71** (1998), 270–278.

[173] V. Shpilrain, *Fixed points of endomorphisms of a free metabelian group*, Math. Proc. Cambridge Phil. Soc. **123** (1998), 77–85.

[174] J. Stallings, *Topology of finite graphs*. Invent. Math. **71** (1983), 551–565.

[175] J. Stallings, *Finiteness properties of matrix representations.* Ann. of Math. (2) **124** (1986), 337–346.

[176] A. Steinberg, *On roots of $a^k b^l$*, Math. Z. **192** (1986), 1–8.

[177] G. Tardos, *On the intersection of subgroups of a free group*, Invent. Math. **108** (1992), 29–36.

[178] G. Tardos, *Towards the Hanna Neumann conjecture using Dicks' method,* Invent. Math. **123** (1996), 95–104.

[179] E. Timoshenko, *Center of a group with one defining relation in the variety of 2-solvable groups* (Russian). Sibirsk. Mat. Z. **14** (1973), 1351–1355, 1368.

[180] E. I. Timoshenko, *Test elements and test rank of a free metabelian group*, Siberian Math. J. **41** (2000), 1200–1204.

[181] E. Ventura, *On fixed subgroups of maximal rank*, Comm. Algebra **25** (1997), 3361–3375.

[182] D. Wise, *A continually descending endomorphism of a finitely generated residually finite group*, Bull. London Math. Soc. **31** (1999), 45–49.

[183] D. Wise, *The residual finiteness of one-relator groups with torsion*, preprint.

[184] P. Wright, *Group presentations and formal deformations*, Trans. Amer. Math. Soc. **208** (1975), 161–169.

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