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ON GENERATORS OF L/R^2 LIE ALGEBRAS

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ABSTRACT. Let L be a free Lie algebra of finite rank n and R its arbitrary ideal. A necessary and sufficient condition for n elements of the Lie algebra L/R^2 to be a generating set is given. In particular, we have a criterion for n elements of a free Lie algebra of rank n to be a generating set which is similar to the corresponding group-theoretic result due to Birman (*An inverse function theorem for free groups*, Proc. Amer. Math. Soc. **41** (1973), 634–638).

1. INTRODUCTION

Let $L = L_n$ be a free Lie algebra of finite rank $n \geq 2$ over the field K of characteristic 0. All background and undefined notions here can be found in [1]. Given an ideal R of L we consider the Lie algebra L/R^2, i.e., a free commutative extension of L/R, and give a necessary and sufficient condition for some n elements of L/R^2 to be its generating set. The similar question for groups has been treated in [4]. Here our main result is the following theorem. Before formulating this theorem we introduce some more notation.

Let $U(L)$ be the universal enveloping algebra of L, i.e., a free associative algebra over the field K. For an ideal R of L, we denote by ε_R the natural homomorphism of L onto L/R. The induced homomorphism of $U(L)$ onto $U(L/R)$ we also denote by ε_R. By $d_i(u)$ we denote the ith Fox derivative of an element $u \in U(L)$ (see [3]).

Theorem. *Let R be an ideal of L, and let $y_1, \dots, y_n$ be elements of L. Then Lie algebra L/R^2 is generated by the images of $y_1, \dots, y_n$ if and only if the matrix $\|d_j(y_i)^{\varepsilon_R}\|_{1\leq i,j\leq n}$ has a left inverse over $U(L/R)$.*

The following corollary is an analog of Birman's result [2] for groups.

Corollary. *A mapping $\phi: x_i \to y_i$, $1 \leq i \leq n$, induces an automorphism of L if and only if the matrix $\|d_j(y_i)\|_{1\leq i,j\leq n}$ has a left inverse over $U(L)$.*

It should be mentioned that a criterion similar to the assertion of this corollary has been obtained recently in [5].

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The arrangement of the paper is as follows. In §2 we present our technical tool—the free differential calculus—and use it in §3 to prove the theorem.

2. Free differential calculus

Let $L = L_n$ be a free Lie algebra over the field K of characteristic 0 with the set $\{x_i\}_{1 \le i \le n}$ of free generators. We write (u, v) for the Lie product of two elements u and v. By Δ we denote the augmentation ideal of its universal enveloping algebra $U(L)$, i.e., the kernel of the augmentation map $\varepsilon\colon U(L) \to K$ defined by $\varepsilon(x_i) = 0$, $1 \le i \le n$. If $R \ne L$ is an ideal of L then by Δ_R we denote the kernel of the natural homomorphism $\varepsilon_R\colon U(L) \to U(L/R)$.

For arbitrary elements $y_1, \dots, y_n, v$ of $U(L)$ we denote by $v[y_1, \dots, y_n]$ the result of the substitution of $y_1, \dots, y_n$ instead of $x_1, \dots, x_n$ in v.

In [3] Fox gave a detailed account of the differential calculus in a free group ring. Since any free associative algebra is naturally embedded in a free group algebra (over the same field), most of the technical results remain valid for free associative algebras.

To be more specific, we introduce here free derivations as the mappings $d_i\colon U(L) \to U(L)$, $1 \le i \le n$, satisfying the following conditions whenever $\alpha, \beta \in K$, $u, v \in U(L)$:

(1) $d_i(x_j) = \delta_{ij}$ (Kronecker delta);
(2) $d_i(\alpha u + \beta v) = \alpha d_i(u) + \beta d_i(v)$;
(3) $d_i(uv) = d_i(u)\varepsilon(v) + u d_i(v)$.

Instead of $d_i(u)$ we sometimes write $\partial u/\partial x_i$.

It is an obvious consequence of the definitions that $d_i(1) = 0$.

It is easy to prove (see [3]) that these derivations have another nature as well. The ideal Δ is a free left $U(L)$-module with free basis $\{x_i\}_{1 \le i \le n}$, and the mappings d_i are projections to the corresponding free cyclic direct summands. Thus any element $u \in \Delta$ can be uniquely written in the form $u = \sum d_i(u) x_i$. Moreover, for any elements $y_1, \dots, y_n$ of $U(L)$ one can always find an element $u \in U(L)$ with $d_i(u) = y_i$, $1 \le i \le n$.

We need some technical lemmas to be used throughout §3. The first lemma is an immediate consequence of the definitions.

Lemma 1. *Let J be an arbitrary ideal of $U(L)$ and let $u \in \Delta$. Then $u \in J\Delta$ if and only if $d_i(u) \in J$ for each i, $1 \le i \le n$.*

The next lemma can be found in [6].

Lemma 2. *Let R be an ideal of L and let $u \in L$. Then $u \in \Delta_R \Delta$ if and only if $u \in R^2$.*

We have the following "chain rule" for Fox derivations (see [3]).

Lemma 3. *Let $y_1, \dots, y_n, w$ be some elements of $U(L)$ and let $v = w[y_1, \dots, y_n]$. Then*

$$d_j(v) = \sum_{1 \le k \le n} \frac{\partial v}{\partial y_k} \cdot \frac{\partial y_k}{\partial x_j}.$$

By $\partial v/\partial y_k$ we mean here $d_k(w)[y_1, \dots, y_n]$.

Finally, we prove the following lemma which seems to be well known although I was not able to find an appropriate reference.

Lemma 4. *Let M be an arbitrary Lie algebra over a field and $U(M)$ its universal enveloping algebra. Let $v_1, \dots, v_m$ and u be elements of M. Suppose u belongs to the left ideal of $U(M)$ generated by $v_1, \dots, v_m$. Then u belongs to the Lie algebra N generated by $v_1, \dots, v_m$.*

Proof. We choose a totally ordered (additive) basis B of M containing a basis of N, such that $u, v \in B$, $u \in N$, $v \notin N \Rightarrow u < v$. Let I be the left ideal of $U(M)$ generated by N. By the Poincarè-Birkhoff-Witt theorem, I is linearly generated by the monomials of the form $w_1 \cdots w_k$, $k \geq 1$, $w_i \in B$, $w_k \in N$. It is easy to check now, using the identity $uv = vu + (u, v)$, that I is linearly generated by the monomials of the form $w_1 \cdots w_k$, $k \geq 1$, $w_i \in B$, $w_k \in N$, $w_1 \geq \cdots \geq w_k$, as well. Applying again the Poincarè-Birkhoff-Witt theorem we see that these decreasing products are linearly independent, so if some s belongs to $M \cap I$ then s is a linear combination of monomials from $B \cap N$, i.e., s is in N.

3. Proof of the theorem

1. We prove first the "only if" part. Let the images of $y_1, \dots, y_n$ generate the Lie algebra L/R^2. Then we can find elements $v_1, \dots, v_n$ of $U(L)$ such that $v_i[y_1, \dots, y_n] = x_i \pmod{R^2}$. Applying Lemmas 3 and 2 we see that the matrix $\|\partial v_i/\partial y_k\|^{\varepsilon_R}_{1 \leq i,k \leq n}$ is left inverse to $\|d_j(y_i)^{\varepsilon_R}\|_{1 \leq i,j \leq n}$ in the ring of matrices with entries from $U(L/R)$.

2. Now we prove the "if" part. Let $J = \|d_j(y_i)^{\varepsilon_R}\|_{1 \leq i,j \leq n}$, and let $\widetilde{X}$ be the column $(\bar{x}_1, \dots, \bar{x}_n)$ and $\widetilde{Y}$ the column $(\bar{y}_1, \dots, \bar{y}_n)$, where we write $\bar{u}$ instead of u^{ε_R}. Then we have $J\widetilde{X} = \widetilde{Y}$, and, multiplying by J^{-1} on the left, $\widetilde{X} = J^{-1}\widetilde{Y}$.

It follows that in the ring $U(L/R)$ the elements $\bar{x}_1, \dots, \bar{x}_n$ belong to the left ideal generated by $\bar{y}_1, \dots, \bar{y}_n$. Hence $\bar{x}_1, \dots, \bar{x}_n$ belong to the Lie algebra generated by $\bar{y}_1, \dots, \bar{y}_n$ in view of Lemma 4, so $\bar{y}_1, \dots, \bar{y}_n$ generate L/R.

Therefore we can find elements $u_1, \dots, u_n$ of $U(L)$ such that

$$\|d_j(u_i)[\bar{y}_1, \dots, \bar{y}_n]\|_{1 \leq i,j \leq n} = J^{-1}.$$

Let $w_i = u_i[y_1, \dots, y_n]$. Clearly we may assume that $w_i^{\varepsilon} = 0$. Then we have by Lemma 3

$$\|d_j(w_i)^{\varepsilon_R}\|_{1 \leq i,j \leq n} = J^{-1}J = I, \tag{1}$$

the identity matrix.

Thus (1) gives $\bar{w}_i = \bar{x}_i$. Since $w_i - x_i \in \Delta_R$, we should have for some $v_j \in R$, $h_j \in U(L)$

$$w_i - x_i = \sum_j v_j h_j. \tag{2}$$

Denote the right-hand side of (2) by S. We prove now that we can replace S with some element t of L with $d_j(S) = d_j(t) \pmod{\Delta_R}$ for any j, $1 \leq j \leq n$.

For, we write

$$S = \sum_j v_j(h_j - h_j^\varepsilon + 1) + \sum_j (h_j^\varepsilon - 1)v_j. \tag{3}$$

The second sum in the right-hand side of (3) is an element of L since each v_j is. Let now $h_j - h_j^\varepsilon + 1 = g_j$. Then

$$d_i(v_j g_j) = d_i(v_j)g_j^\varepsilon + v_j d_i(g_j).$$

But $g_j^\varepsilon = 1$, and $v_j \in R$, so

$$d_i(v_j g_j) = d_i(v_j) \pmod{\Delta_R}.$$

Hence $d_i(S) = d_i(t) \pmod{\Delta_R}$, where

$$t = \sum_j v_j + \sum_j (h_j^\varepsilon - 1)v_j = \sum_j h_j^\varepsilon v_j.$$

Now we take $\hat{u}_i = x_i + \sum_j h_j^\varepsilon v_j$ instead of u_i and $\hat{w}_i = \hat{u}_i[y_1, \dots, y_n]$. Then

$$\|d_j(\hat{u}_i)[\bar{y}_1, \dots, \bar{y}_n]\|_{1\le i,j\le n} = J^{-1}$$

since $d_j(\hat{u}_i) = d_j(u_i) \pmod{\Delta_R}$. It follows that

$$\|d_j(\hat{w}_i)^{\varepsilon_R}\|_{1\le i,j\le n} = J^{-1}J = I$$

in view of Lemma 3. This means that $d_j(\hat{w}_i)^{\varepsilon_R} = \delta_{ij}$, i.e.,

$$d_i(\hat{w}_i) - 1 \in \Delta_R, \qquad d_j(\hat{w}_i) \in \Delta_R \quad \text{for } j \neq i.$$

Hence

$$d_i(\hat{w}_i)x_i - x_i \in \Delta_R\Delta \quad \text{and} \quad d_j(\hat{w}_i)x_j \in \Delta_R\Delta \qquad \text{for } j \neq i. \tag{4}$$

Now take a sum of all the inclusions (4) for a fixed i. This yields $\hat{w}_i - x_i \in \Delta_R\Delta$ in view of $u = \sum d_i(u)x_i$ for any $u \in U(L)$. Hence, as $\hat{w}_i - x_i \in L$, Lemma 2 gives $\hat{w}_i - x_i \in R^2$, or $\hat{w}_i = x_i \pmod{R^2}$, and it follows that the natural images of $y_1, \dots, y_n$ generate L/R^2. The theorem is proved.

Now taking $R = \{0\}$ we obtain the following.

Corollary. *A mapping $\phi: x_i \to y_i$, $1 \le i \le n$, induces an automorphism of L if and only if the matrix $\|d_j(y_i)\|_{1\le i,j\le n}$ has a left inverse over $U(L)$.*

Remark. The analog of this corollary does not hold in a free associative algebra. Indeed, consider the free associative algebra F over a field of characteristic 0 with free generators x_1 and x_2. Take the mapping $\phi: x_1 \to x_1 + x_1x_2$, $x_2 \to x_2$. Then J_ϕ is invertible in the ring of 2×2 matrices with the entries from F, while ϕ is not an automorphism of F.

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