REPRESENTATIONS AND RIGIDITY OF Aut(F_3)

STEPHEN P. HUMPHRIES

ABSTRACT. We construct some new matrix representations of Aut(F_3) and show that there are infinitely many distinct irreducible representations of Aut(F_3) in dimensions at most 42.

§1. Introduction

Let $F_n =< x_1, \ldots, x_n >$ be the free group of rank n and let Aut(F_n) denote the group of automorphisms of F_n, where our convention is that Aut(F_n) acts on the right. Note that Cox(B_n), the Coxeter group of type B_n, is naturally a subgroup of Aut(F_n) and that Cox(B_n) $\subset$ Aut(F_n) is generated by elements P, Q, S where

$$P : x_1 \mapsto x_2, \;\; x_2 \mapsto x_1, \quad x_i \mapsto x_i, i > 2;$$
$$Q : x_i \mapsto x_{i+1}, \;\; i < n, \quad x_n \mapsto x_1;$$
$$S : x_1 \mapsto x_1^{-1}, \;\; x_i \mapsto x_i, \;\; i > 1.$$

Then Nielsen [Ni] has given a presentation of Aut(F_n) with generators P, Q, S, U where P, Q, S are as above and

$$U : x_1 \mapsto x_1 x_2, \;\; x_i \mapsto x_i, \;\; i > 1.$$

The relations needed are [MKS, Ni]:

$$P^2, \quad (QP)^{n-1}, \;\; Q^n,$$
$$[P, Q^{-i}PQ^i], \;\; i = 2, \ldots, \lfloor n/2 \rfloor,$$
$$S^2, \;\; [S, Q^{-1}PQ], \;\; [S, QP], \;\; [S, Q^{-1}SQ],$$
$$(PSPU)^2, \;\; U^{-1}PUPSUSPS,$$
$$(PQ^{-1}UQ)^2 UQ^{-1}U^{-1}QU^{-1},$$
$$[U, Q^{-2}PQ^2], \;\; [U, QPQ^{-1}PQ],$$
$$[U, Q^{-2}SQ^2], \;\; [U, Q^{-2}UQ^2],$$
$$[U, SUS], \;\; [U, PQ^{-1}SUSQP],$$
$$[U, PQ^{-1}PQPUPQ^{-1}PQP].$$

Here $[X, Y] = XYX^{-1}Y^{-1}$.

In fact the generator P is not necessary. To see this one need only consider the case $n = 3$, where one can check that

$$P = SUSQUQSUQ^{-1}U^{-1}QU^{-1}QU.$$

Here $P, Q, S, U \in \mathrm{Aut}(F_3)$. Neumann [Ne] also gave a generating set with three elements.

A group is said to be *rigid* if it has only finitely many irreducible representations over the complex numbers of each degree. In the list of problems on group theory that can be found at [O] it is asked whether $\mathrm{Aut}(F_n)$ is rigid. In [H] we showed that braid groups $B_n, n > 2$, are not rigid and showed that it then followed that $\mathrm{Aut}(F_2)$ is not rigid. Also see [BV, PR] for other results on similar notions of rigidity. The result for braid groups is closely connected to the fact that there are representations $\rho : B_n \to GL(n, \mathbb{Q}[t, t^{-1}])$ such that $\mathrm{trace}(\rho(g))$ is a non-trivial rational function of t for some $g \in B_n$; for example consider the Burau representation [Bir].

In this paper we construct a representation $\rho : \mathrm{Aut}(F_3) \to GL(42, \mathbb{Q}[t_1, t_1^{-1}])$ having the property that $\mathrm{trace}(\rho(U^2))$ is a non-trivial rational function of the t_i. We will then indicate how this shows that $\mathrm{Aut}(F_n)$ is not rigid.

The representation

Let G be a group and let H be a subgroup of finite index k. Let $\rho : H \to GL(m, R)$ be a linear representation of H, where R is a commutative ring with identity. Then the induced representation is constructed as follows: Let $Hx_1, \dots, Hx_k$ be the cosets of H and define $\rho_H^G : G \to GL(km, R)$ where for $g \in G$ we let

$$\rho_H^G(g) = \begin{pmatrix} \rho(x_1gx_1^{-1}) & \rho(x_1gx_2^{-1}) & \dots & \rho(x_1gx_k^{-1}) \\ \rho(x_2gx_1^{-1}) & \rho(x_2gx_2^{-1}) & \dots & \rho(x_2gx_k^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(x_ngx_1^{-1}) & \rho(x_ngx_2^{-1}) & \dots & \rho(x_ngx_k^{-1}) \end{pmatrix}.$$

Here the convention is that $\rho(y)$ is the 0 matrix if $y \notin H$. Then ρ_H^G is a representation of G [Sc].

Let N denote the normal closure of U^2 in $\mathrm{Aut}(F_3)$. Then one can check using Magma [Ma] that N has index 2688 in $\mathrm{Aut}(F_3)$. It can also be checked that the abelianisation $\mathrm{Ab}(N)$ is isomorphic to $\mathbb{Z}_2^3 \times \mathbb{Z}^{14}$.

Now define the subgroup

$$\begin{aligned} H &=< N, S, SPSQ^{-1}UPUQ^{-1}UQ^{-1}, PQ, SPSUQUPU, PSP, QSQ^{-1} > \\ &=< SPSUQUPU, PSP, QSQ^{-1}, SPSQ^{-1}UPUQ^{-1}UQ^{-1}, UPU^{-2}PU^{-1}, PQ > . \end{aligned}$$

Then one finds that H has index 42 in $\mathrm{Aut}(F_3)$ and that $\mathrm{Ab}(H) \cong \mathbb{Z}_2^5 \times \mathbb{Z}$. Let

$$\rho : H \to \mathrm{Ab}(H) \to \mathbb{Z} =< t >$$

be the projection. Let $P' = \rho_H^G(P), Q' = \rho_H^G(Q)$ etc. Then the following facts can be checked [Ma]:

(1) the minimal polynomial of U' is

$$(x^2 - 1)(x^2 - t)(x^2 - t^{-1}).$$

(2) $\operatorname{trace}((U')^2) = (4t^2 + 34t + 4)/t$.
(3) The representation ρ_H^G is irreducible (put $t = i \in \mathbb{C}, i^2 = -1$ and you get an irreducible finite representation).
(4) If we replace t by t^2, then the resulting matrix $U'(t^2)$ is diagonalisable.
(5) Replacing t by a kth root of unity ensures that U' has order a divisor of $2k$.

Fact (5) is enough to see that there are infinitely many distinct irreducible representations of Aut(F_3) of dimension at most 42.

One can repeat the above construction using the subgroup

$$K =< N, U, PSQ^{-1}UQ >$$

which has index 56. The same conclusions (1)-(5) are true in this case also.

Appendix

The matrices P', Q', S', U' are too large to give here. The interested reader can construct them easily using any of the algebra systems Magma, GAP, Magnus etc. We here provide the code to generate these matrices in Magma [Ma]; the matrices are called PP, QQ, SS, UU:

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A<P,Q,S,U>:=Group<P,Q,S,U|
                        P^2,Q^3,S^2,(Q*P)^2,S*Q*P=Q*P*S,
                        S*Q^-1*P*Q=Q^-1*P*Q*S,S*Q^-1*S*Q=Q^-1*S*Q*S,
                        (U,Q*P*Q^-1*P*Q),
                        (U,Q^-2*S*Q^2),(P*S*P*U)^2,
                        P*U*P*S*U*S*P*S=U,
                        (P*Q^-1*U*Q)^2*U*Q^-1=U*Q^-1*U,
                        U*S*U*S=S*U*S*U,
                        (U,P*Q^-1*S*U*S*Q*P),
                        (U,P*Q^-1*P*Q*P*U*P*Q^-1*P*Q*P)>;

N:=NormalClosure(A,sub<A|U^2>);
H:=sub<A|N,S,
 S * P * S * Q^-1 * U * P * U * Q^-1 * U * Q^-1,
 P * Q,S * P * S * U * Q * U * P * U,
 P * S * P,Q * S * Q^-1>;
Index(A,H);
rH1,rH2:=Rewrite(A,H);
a1,a2:=AbelianQuotient(rH1);
aq:=AQInvariants(rH1);
Z:=AbelianGroup([0]);
ga:=[Z|0,0,0,0,0];
for i:=1 to 1 do
 ga:=ga cat [Z.i];
end for;
ph:=hom<a1 -> Z|ga>;
R:=RationalField();
Po<p>:=RationalFunctionField(R,1);
t1,t2:=Transversal(A,H);
t1:=SetToSequence(t1);
mrr:=MatrixRing(Po,#t1);
sm:=[];
for ii:=1 to  #Generators(A) do
 m:=mrr!0;
 for i,j in [1..#t1] do
  if t1[i]*A.ii*t1[j]^-1 in H then
      m[i,j]:=p^ElementToSequence
             (ph(a2((rH1!((t1[i]*A.ii*t1[j]^-1))))))[1];
  end if;
 end for;
 sm:=sm cat [m];
end for;

PP:=sm[1];QQ:=sm[2];SS:=sm[3];UU:=sm[4];
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References

[Bir] Birman, Joan, *Braids, Links and Mapping class Groups*, Ann. Math. Studies **82** (1974).

[BV] Bridson, Martin R. Vogtmann, Karen, *Homomorphisms from automorphism groups of free groups*, Bull. London Math. Soc. **35** (2003), 785-792.

[H] Humphries, Stephen P., *Braid groups and Aut*(F_2) *are not rigid*, Group theory, statistics, and cryptography, Contemp. Math., vol. 360, Amer. Math. Soc., Providence, RI, 2004, pp. 51-54.

[Ma] Bosma W., Cannon J., *MAGMA (University of Sydney)*, 1994.

[MKS] Magnus W., Karrass A., Solitar D., *Combinatorial Group Theory*, Dover, 1976.

[Ne] Neumann B.H., *Die Isomorphismengruppe der freien Gruppen*, Math. Ann. **107**, 367-386.

[Ni] Nielsen, J., *Die Isomorphismengruppe der freien Gruppen*, Math. Ann. **91** (1924), 169-209.

[O] *http://www.grouptheory.info/*.

[PR] Potapchik, A.; Rapinchuk, A., *Low-dimensional linear representations of* $Aut\, F_n, n \geq 3.$, Trans. Amer. Math. Soc. **352** (2000), 1437-1451.

[Sc] Scott W. R., *Group Theory*, Dover, 1987.