Nielsen's commutator test for two-generator groups

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1 Introduction

Nielsen [14] gave the following commutator test for an endomorphism of the free group $F = F_2 = \langle x, y; \varnothing \rangle$ to be an automorphism: an endomorphism $\phi\colon F \to F$ is an automorphism if and only if the commutator $[\phi(x), \phi(y)]$ is conjugate in F to $[x, y]^{\pm 1}$. He obtained this test as a corollary to his well-known result that every IA-automorphism of F (i.e. one which fixes F modulo its commutator subgroup) is an inner automorphism. Bachmuth *et al.* [4] have proved that IA-automorphism of most two-generator groups of the type F/R' are inner, and it becomes natural to ask if Nielsen's commutator test remains valid for those groups as well. Durnev [6] considered this question for the free metabelian group F/F'' and confirmed the validity of the commutator test in this case. Here we prove that Nielsen's test does not hold for a large class of F/R' groups (Theorem 3.1) and, as a corollary, deduce that it does not hold for any non-metabelian solvable group of the form F/R'' (Corollary 3.2). In view of our Theorem 3.1, Nielsen's commutator test in these situations seems to have less appeal than his result that the IA-automorphisms of F are precisely the inner automorphisms of F. We explore some applications of this important result with respect to non-tameness of automorphisms of certain two-generator groups F/R (i.e. automorphisms of F/R which are not induced by those of the free group F). For instance, we show that a two-generator free polynilpotent group F/V, $V \neq F$, has non-tame automorphisms except when $V = \gamma_2(F)$ or $V = \gamma_3(F)$, or when V is of the form $[\gamma_n(U), \gamma_n(U)]$, $n \geqslant 2$ (Theorem 4.2). This complements the results of [8] and [16] rather nicely, and it shown to follow from a more general result (Proposition 4.1). We also include an example of an endomorphism $\theta\colon x \to xu,\ y \to y$ of F which induces a non-tame automorphism of $F/\gamma_6(F)$ while the partial derivative $\partial(u)/\partial(x)$ is "balanced" in the sense of Bryant *et al.* [5] (Example 4.4). This gives an alternative solution of a problem in [5] which has already been resolved by Papistas [15] in the negative. In our final section, we consider groups of the type $F/[R', F]$ and, in contrast to groups of the type F/R', we show that the Nielsen's commutator test does hold in most of these groups (Theorem 5.1). We conclude with a sufficiency condition under which Nielsen's commutator test is valid for a given pair of generating elements of F modulo $[R', F]$ (Proposition 5.2).

2 Notation and preliminaries

We use standard language of free group rings. For any unexplained notation or terminology the reader may consult [9], Chapter I. Let $F = \langle x, y; \varnothing \rangle$ be the free group on x and y, and let R be a normal subgroup of F. There is a familiar action via conjugation of the group ring $\mathbb{Z}(F/R)$ on the abelian group R/R' under which R/R' becomes the relation module for F/R. Accordingly, there is a map $(F/R, R/R') \to R/R'$ defined by $(fR, rR') \to (f^{-1}rf)R'$, and linearly extended to $(\mathbb{Z}(F/R), R/R') \to R/R'$. Where the meaning is clear from the context, we denote the elements of $\mathbb{Z}F$ and their natural images in $\mathbb{Z}(F/R)$ by the same letters without ambiguity. Thus, for any $v \equiv \sum_i n_i f_i \bmod \mathbb{Z}F(R-1)$, $n_i \in \mathbb{Z}$, $f_i \in F$ and $r \in R$, the result of the action of $\sum_i n_i f_i R \in \mathbb{Z}(F/R)$ on rR' is denoted by $r^v R'$.

We denote by $\Delta_F = \mathbb{Z}F(F-1)$ and $\Delta_R = \mathbb{Z}F(R-1)$ the augmentation ideals in $\mathbb{Z}F$. Since Δ_F is a free left $\mathbb{Z}F$-module (as well as a free right $\mathbb{Z}F$-module) with basis $\{(x-1), (y-1)\}$ (see [9, page 5]), every element $z \in \Delta_F$ can be written uniquely as

$$z = \partial z/\partial x(x-1) + \partial z/\partial y(y-1),$$

where $\partial z/\partial x, \partial z/\partial y \in \mathbb{Z}F$ are known as Fox (left) derivatives (cf. [7]). Similarly, every element $z \in \Delta_F$ can be written uniquely as

$$z = (x-1)\partial' z/\partial' x + (y-1)\partial' z/\partial' y,$$

where $\partial' z/\partial' x, \partial' z/\partial' y \in \mathbb{Z}F$ are known as Fox (right) derivatives. We gather some necessary facts about these derivatives in the following lemma. The general reference is [9, Chapter I].

Lemma 2.1.

(a) *Let J be an arbitrary ideal of $\mathbb{Z}F$ and let $u \in \Delta_F$. Then $u \in J\Delta_F$ if and only if $\partial u/\partial x \in J$ and $\partial u/\partial y \in J$;*

(b) *If $w \in \gamma_c(F)$, $c \geqslant 2$, then $\partial w/\partial x, \partial w/\partial y \in (\Delta_F)^{c-1}$;*

(c) *Let $r \in R$ and $v \in \mathbb{Z}F \setminus \Delta_R$. If $r^v \equiv 1 \pmod{R'}$ then $v^*\partial r/\partial x \equiv 0 \pmod{\Delta_R}$ and $v^*\partial r/\partial y \equiv 0 \pmod{\Delta_R}$, where for $v = \sum_i n_i f_i$, $v^* = \sum_i n_i f_i^{-1}$.*

[Since, modulo $\Delta_R\Delta_F$, $r^f - 1 \equiv f^{-1}(r-1)$ for any $f \in F$, the proof of (c) follows from $F \cap (1 + \Delta_R\Delta_F) = R'$ using (a).]

Consider now an automorphism τ of F of the form

$$\tau\colon x \to xr,\ y \to ys \ \text{ where } \ r, s \in F'.$$

Then τ is an IA-automorphism and hence, by Nielsen's characterization, τ must be an inner automorphism $x \to x^f$, $y \to y^f$, induced by some $f \in F$. Comparing the two formulations of τ instantly yields:

Lemma 2.2. *The mapping $\tau\colon x \to xr,\ y \to ys$ with $r, s \in F'$ defines an automorphism of F if and only if $r = [x, f]$ and $s = [y, f]$ for some $f \in F$. Similarly, the mapping $\tau'\colon x \to rx,\ y \to sy$ with $r, s \in F'$ defines an automorphism of F if and only if $r = [f, x^{-1}]$ and $s = [f, y^{-1}]$ for some $f \in F$.*

For each $c \geqslant 3$, the IA-endomorphism $\tau\colon x \to x[x, y, y],\ y \to y$ of F induces an automorphism of the free nilpotent group $F/\gamma_{c+1}(F)$ of class c. If τ induces a tame automorphism of $F/\gamma_{c+1}(F)$ (i.e. one induced by an automorphism of F) then, for some $r, s \in \gamma_{c+1}(F)$, the endomorphism $\tau'\colon x \to x[x, y, y]r,\ y \to ys$, must be an automorphism of F. By Lemma 2.2, we must then have $[x, y, y]r = [x, f]$ and $s = [y, f]$ for some $f \in F$. In particular $f \in F'$. It is well-known fact that if $u \in \gamma_i(F)$, $u \notin \gamma_{i+1}(F)$ and $v \in \gamma_j(F)$, $v \notin \gamma_{j+1}(F)$ for some $i \neq j$ then $[u, v] \notin \gamma_{i+j+1}(F)$. Since $s = [y, f] \in \gamma_{c+1}(F)$, it follows that $f \in \gamma_c(F)$ and in turn $[x, y, y] \in \gamma_4(F)$, which is not the case. We thus have the following corollary to Lemma 2.2.

Corollary 2.3 (cf. [1, 3])**.** *A two-generator free nilpotent group of class $c \geqslant 3$ has non-tame automorphisms.*

3 Nielsen's commutator test

Let $R < F'$ be a normal subgroup of the free group F. In this section, we elaborate on some examples of groups of the form F/R' for which Nielsen's commutator test is not valid. Specifically, we prove

Theorem 3.1. *Let $R < F'$ be a non-trivial normal subgroup of F such that*

(i) *the centre of $F/[R, F]$ is $R/[R, F]$ and*

(ii) *the group ring $\mathbb{Z}(F/R)$ is an Ore domain (i.e. $\mathbb{Z}(F/R)$ has no zero divisors and any two non-zero elements have a common non-zero multiple).*

Then there exist $u, v \in F$ such that $[u, v] \equiv [x, y] \pmod{R'}$, while the endomorphism $x \to u,\ y \to v$ of F does not induce an automorphism of F/R'.

Proof. Choose some $r \in R$, $r \notin R'$. Let $u \equiv r^a x \pmod{R'}$ and $v \equiv r^b y \pmod{R'}$ for some $a, b \in \mathbb{Z}F$ (to be specified later). Then, modulo R' we have

$$\begin{aligned}[r^a x, r^b, y] &\equiv [r^a x, y][r^a x, r^b]^y \equiv [r^a, y]^x [x, y][x, r^b]^y \equiv [x, y][r^a, y]^{x[x,y]}[x, r^b]^y \\ &\equiv [x, y][r^a, y]^{y^{-1}xy}[x, r^b]^y \equiv [x, y][y^{-1}, r^a]^{xy}[r^b, x^{-1}]^{xy} \\ &\equiv [x, y] r^{(a(1-y^{-1})+b(x^{-1}-1))xy}.\end{aligned}$$

Thus, the congruence $[r^a x, r^b y] \equiv [x, y] \pmod{R'}$ is equivalent to the congruence

$$r^{(a(1-y^{-1})+b(x^{-1}-1))} \equiv 1 \pmod{R'}.$$

Since $\mathbb{Z}(F/R)$ has no zero-divisors, it follows by Lemma 2.1 that

$$a(y^{-1} - 1) \equiv b(x^{-1} - 1) \text{ modulo } \Delta_R, \tag{1}$$

where $a, b \in \Delta_F$ (since $R < F'$ implies that Δ_R lies in Δ_F^2). By condition (ii) of our theorem there exist $a, b \notin \Delta_R$ satisfying the congruence (1). We choose one such pair of elements a and b satisfying (1) and consider elements u and v of F such that

$$u \equiv r^a x \quad \text{and} \quad v \equiv r^b y \pmod{R'}.$$

For this choice of u and v, it is easy to see that the congruence $[u, v] \equiv [x, y] \pmod{R'}$ holds. Suppose that the mapping $x \to u,\ y \to v$ induces an automorphism of the group F/R'. Then this being an IA-automorphism of F/R' it must be inner by a result of Bachmuth *et al.* [4]. In that case, as in Lemma 2.2, we would have, for some $f \in F$,

$$r^a \equiv [f, x^{-1}] \pmod{R'} \quad \text{and} \quad r^b \equiv [f, y^{-1}] \pmod{R'}. \tag{2}$$

Since r^a, r^b are in $[R, F]$ $(a, b \in \Delta_F)$, it follows that $f[R, F]$ is in the centre of $F/[R, F]$ which by hypothesis is $R/[R, F]$. Thus $f \in R \setminus R'$. Now, (2) yields, by commuting appropriately with y^{-1} and x^{-1}, the congruences

$$r^{a(y^{-1}-1)} \equiv [f, x^{-1}, y^{-1}] \quad \text{and} \quad r^{b(x^{-1}-1)} \equiv [f, y^{-1}, x^{-1}] \pmod{R'},$$

which upon using (1) yield the congruence

$$[f, x^{-1}, y^{-1}] \equiv [f, y^{-1}, x^{-1}] \pmod{R'},$$

or, equivalently, $[f, x^{-1}y^{-1}] \equiv [f, y^{-1}x^{-1}] \pmod{R'}$, since $f \in R$. Now, an application of this last congruence in the expansion of the equation $[f, x^{-1}y^{-1}] = [f, [x, y]y^{-1}x^{-1}]$ modulo R' gives $[f, [x, y]] \in R'$ which implies (using Lemma 2.1) that $[y, x] \in R$ implying that F/R is abelian, contrary to the choice of R. This completes the proof of the theorem. □

The following corollary to Theorem 3.1 sharply contrasts the cited result of Durnev [6].

Corollary 3.2. *Let N be a proper normal subgroup of F such that F/N is solvable. Then there are elements u, v in F such that $[u, v] \equiv [x, y] \pmod{N''}$ but the endomorphism $x \to u,\ y \to v$ of F does not induce an automorphism of F/N''.*

Proof. Since F/N' is a torsion-free solvable group (see [9, p. 23]), it follows by a recent theorem of Kropholler *et al.* [11] that the group ring $\mathbb{Z}(F/N')$ has no zero divisors. Thus with $R = N'$, F/R is solvable and $\mathbb{Z}(F/R)$ has no zero divisors. It follows by a result of Lewin [12] that $\mathbb{Z}(F/R)$ is an Ore domain. Since the centre of $F/[R, F]$ $(= F/[N', F])$ is always $R/[R, F]$ (see [9, p. 117]), the hypothesis of Theorem 3.1 is satisfied by R and the corollary follows. □

Corollary 3.3.

(a) *For each $k \geqslant 3$, there exists an endomorphism $\phi_k \colon F \to F$ which does not induce an automorphism of the free solvable group $F/F^{(k)}$ of derived length k, whereas (by Corollary 3.2) $[\phi_k(x), \phi_k(y)] \equiv [x, y] \bmod F^{(k)}$;*

(b) (b) *For each* $c \geqslant 3$*, there exists an endomorphism* $\psi_c\colon F \to F$ *which does not induce an automorphism of* $F/[\gamma_c(F), \gamma_c(F)]$*, whereas* $[\psi_c(x), \psi_c(y)] \equiv [x, y] \bmod [\gamma_c(F), \gamma_c(F)]$ *(by Theorem* 3.1*).*

4 Non-tameness of $\mathrm{Aut}(F/V)$

We recall a result of Bachmuth *et al.* [4] which states that if R is a normal subgroup of the free group $F = \langle x, y; \varnothing\rangle$ such that R is contained in the commutator subgroup F' and the group ring $\mathbb{Z}(F/R)$ has no zero divisors, then every IA-automorphism of the group F/R' is an inner automorphism and hence, in particular, is tame. This phenomenon seems to be limited only to F/R' groups as we prove

Proposition 4.1. *Let* U *and* V *be fully invariant non-trivial subgroups of* F *such that* F/U *is infinite and* $[U', F] \geqslant V \geqslant \gamma_m(U)$ *for some* $m \geqslant 3$*. Then the group* F/V *has non-tame IA-automorphism.*

Proof. Choose some $u \in U'$, $u \notin [U', F]$. Then the mapping $\tau\colon x \to xu,\ y \to y$ induces an automorphism of the group F/V (since U/V is nilpotent). If $\tau\colon x \to xu,\ y \to y$ induces a tame automorphism of F/V then by Lemma 2.2 we would have $u = [x, f]r$ and $1 = [y, f]s$ for some $f \in F$ and $r, s \in V$. In particular, $[x, f] \in U'$ and $[y, f] \in U'$ which implies fU' lies in the centre of F/U'. Since F/U is infinite, the centre of F/U' is trivial ([2], see [9, p. 26]). Thus $f \in U'$ which in turn implies $u \in [U', F]$, contrary to the choice of u. This completes the proof. □

If F/V is a non-nilpotent free polynilpotent group other than of the form $F/[U, U]$, then there is a fully invariant subgroup U with F/U infinite such that for some $m \geqslant 3$, $[U', F] \geqslant V = \gamma_m(U)$. Thus together with Corollary 2.3 and the fact that every IA-automorphism of $F/[\gamma_n(W), \gamma_n(W)]$, $n \geqslant 2$, is inner (see [4]), Proposition 4.1 immediately yields the following:

Theorem 4.2. *Let* F/V *be a non-trivial free polynilpotent group of rank two. Then* F/V *has non-tame IA-automorphisms except when* $V = \gamma_2(F)$ *or* $V = \gamma_3(F)$*, or when* V *is of the form* $[\gamma_n(W), \gamma_n(W)]$*,* $n \geqslant 2$*.*

The requirement in Proposition 4.1 that F/U be infinite can be relaxed as we prove:

Proposition 4.3. *Let* U *and* V *be non-trivial fully invariant subgroups of* F *such that* $[\gamma_3(U), F] \geqslant V \geqslant \gamma_m(U)$ *for some* $m \geqslant 4$*. Then the group* F/V *has non-tame automorphisms.*

Proof. Let c be maximal with respect to the property that $\gamma_3(U) \leqslant \gamma_c(F)$. Then $c \geqslant 3$. We choose $u \in \gamma_3(U) \setminus \gamma_{c+1}(F)$. Assume that the endomorphism $\tau\colon x \to xu,\ y \to y$ induces a tame automorphism of the group F/V. Then, by Lemma 2.2, for some f in F we must have $u = [x, f]v_1$, $1 = [y, f]v_2$ for some $v_1, v_2 \in V \leqslant [\gamma_3(U), F] \leqslant \gamma_{c+1}(F)$. Since, $[x, f], [y, f] \in \gamma_c(F)$, it follows that

$f \in \gamma_{c-1}(F)$. Further, $[y, f] \in \gamma_{c+1}(F)$ implies, as before, that $f \in \gamma_c(F)$. Now $u = [x, f]v_1$ implies that $u \in \gamma_{c+1}(F)$, contrary to the choice of u. □

Remark. Results similar to those in Theorem 4.2 can also be proved easily for some other relatively free groups defined by outer commutator words (e.g. $F/[\gamma_n(F), \gamma_m(F)]$, $n > m \geqslant 2$). We omit the details.

We conclude this section by answering a question from Bryant *et al.* [5]. Following [5], we call an element z of $(\Delta_F)^m$ "balanced" if, modulo $(\Delta_F)^{m+1}$, z belongs to the additive subgroup of $(\Delta_F)^m$ spanned by all elements of the form

$$(x_{i(1)} - 1)(x_{i(2)} - 1)\dots(x_{i(m)} - 1) - (x_{i(2)} - 1)\dots(x_{i(m)} - 1)(x_{i(1)} - 1),$$

where $x_{i(j)} \in \{x, y\}$. The main result of [5] is a necessary condition for tameness of certain automorphisms of free nilpotent groups in terms of certain sums of partial derivatives being balanced, and it is asked in [5] if the condition is also sufficient. Adapted to the rank two situation the criterion states that if an automorphism ϕ of the group $F/\gamma_{c+1}(F)$, $c \geqslant 3$, induces by $x \to xu,\ y \to yv$ with $u, v \in \gamma_c(F)$, is tame then $\partial u/\partial x + \partial v/\partial y$ must be a balanced element of $(\Delta_F)^{c-1}$. Papistas [15] has recently answered the above question in the negative by proving that the condition of being balanced is not in general sufficient. Here we offer an alternative proof.

Example 4.4. Consider the automorphism of the group $F/\gamma_6(F)$ induced by the following endomorphism of F:

$$\phi\colon x \to xu,\ y \to y \ \text{ with } \ u = [[x, y], [x, y, y]].$$

If ϕ induces a tame automorphism of $F/\gamma_6(F)$ then, by Lemma 2.2, we must have $r, s \in \gamma_6(F)$ and $f \in F$ such that $ur = [x, f]$ and $s = [y, f]$ holds in F. This implies, in particular, that $[f, x], [f, y] \in F''\gamma_6(F)$ so that $fF''\gamma_6(F)$ is in the centre of $F/F''\gamma_6(F)$ which is known to be $F''\gamma_5(F)/F''\gamma_6(F)$ (see [13, 36·22]). Thus $f \in F''\gamma_5(F) \leqslant \gamma_5(F)$ (since $F'' < \gamma_5(F)$ for F of rank 2). But then $u = [x, f]r^{-1}$ belongs to $\gamma_6(F)$, contrary to hypothesis. Thus ϕ does not induce a tame automorphism of $F/\gamma_6(F)$.

For any $r, s \in \gamma_6(F)$, we next exhibit $\partial(ur)/\partial x + \partial s/\partial y$ modulo $(\Delta_F)^5$ and show that it is balanced. Indeed, working modulo $(\Delta_F)^5$ we have

$$\partial r/\partial y \equiv 0, \quad \partial s/\partial y \equiv 0,$$

and

$$\partial[[x, y], [x, y, y]]/\partial x \equiv (y-1)^2(x-1)(y-1) + 2(x-1)(y-1)^3 - 3(y-1)(x-1)(y-1)^2$$

which is clearly a balanced element of $(\Delta_F)^4$.

Remark. With the choice of $u = [[x, y], [x, y, \dots, y]] \in \gamma_c(F) \cap F''$, $u \notin \gamma_{c+1}(F)$, $c \geqslant 6$, a similar argument yields examples for the higher values of c. We omit the details.

5 Commutator test for groups of type $F'/[R', F]$

While Nielsen's commutator test does not hold for most F/R' groups of rank 2, the groups of the form $F/[R', F]$ behave altogether differently. Here we prove the following result.

Theorem 5.1. *Let R be a non-trivial normal subgroup of the free group $F = \langle x, y; \varnothing\rangle$. If $[u, v] \equiv [x, y]^{\pm g} \bmod [R', F]$ for some $g \in F$, then u and v generate F modulo $[R', F]$.*

Proof. We may clearly assume that $[u, v] \equiv [x, y] \bmod [R', F]$. Since $[R', F]-1 \leqslant \Delta_F\Delta_R\Delta_F$ (see, for instance, [9, p. 113]), we have:

$$[u, v] \equiv [x, y] \bmod \Delta_F\Delta_R\Delta_F. \tag{3}$$

Taking right Fox derivatives of both sides of (3) yields

$$\left.\begin{aligned} \partial' u/\partial' x(v - [u, v]) + \partial' v/\partial' x(1 - v^{-1}uv) &\equiv (y - [x, y]) \bmod \Delta_R\Delta_F \\ \partial' u/\partial' y(v - [u, v]) + \partial' v/\partial' y(1 - v^{-1}uv) &\equiv (1 - y^{-1}xy) \bmod \Delta_R\Delta_F. \end{aligned}\right\} \tag{4}$$

Now, taking the left Fox derivatives of both sides of the congruences in (4) yield four congruences modulo Δ_R given by the following matrix equation:

$$\begin{pmatrix} \partial' u/\partial' x & \partial' v/\partial' x \\ \partial' u/\partial' y & \partial' v/\partial' y \end{pmatrix} \begin{pmatrix} \partial(v - [u, v])/\partial x & \partial(v - [u, v])/\partial y \\ \partial(1 - v^{-1}uv)/\partial x & \partial(1 - v^{-1}uv)/\partial y \end{pmatrix}$$
$$= \begin{pmatrix} x^{-1}y^{-1}(y - 1) & 1 - x^{-1}y^{-1}(x - 1) \\ -y^{-1} & y^{-1} - y^{-1}x \end{pmatrix}.$$

It is easily verified that the matrix on the right-hand side above is invertible over $\mathbb{Z}F$, and hence also over $\mathbb{Z}F \pmod{\Delta_R}$. Hence the Jacobian matrix on the left-hand side is also invertible over $\mathbb{Z}F \pmod{\Delta_R}$ which, by a result of Krasnikov [10] (see [9, p. 29]), implies that u and v generate F modulo R'. It follows that u and v also generate F modulo any normal subgroup V of R such that R/V is nilpotent. Thus u and v generate F modulo $[R', F]$, as was to be proved. □

Although the elements u and v generate the group $F/[R', F]$ under the conditions of Theorem 5.1 we cannot, in general, conclude that the mapping induced by $x \to u,\ y \to v$ defines an automorphism of the group $F/[R', F]$ since, for instance, $F/[R', F]$ may be non-Hopfian. Since $[R', F]$ might not be fully invariant in F, this mapping may not even define an endomorphism of $F/[R', F]$. As a (partial) converse of Theorem 5.1, we can prove the following result.

Proposition 5.2. *Let R be a normal subgroup of the free group F such that $R \leqslant F'$ and the group ring $\mathbb{Z}(F/R)$ has no zero divisors. If the mapping $\tau\colon x \to u,\ y \to v$ induces an automorphism of the group $F/[R', F]$ then $[u, v] \equiv [x, y]^{\pm g} \bmod [R', F]$ for some $g \in F$.*

Proof. Let τ induce some automorphism ψ of the group F/R'. Then ψ is tame (Bachmuth *et al.* [4]) and hence a composition of the inner automorphisms of F/R' together with the automorphisms of F/R' induced by the maps $x \to y,\ y \to x$ and $x \to xy,\ y \to y$. It suffices, therefore, to verify that the congruence $[\tau(x), \tau(y)] \equiv [x, y]^{\pm g} \bmod [R', F]$ holds for some g in F when τ is assumed to be an inner automorphism of $F/[R', F]$, or else defined by the maps $x \to y,\ y \to x$ and $x \to xy,\ y \to y$. These verifications are straightforward and we omit the details. □

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