SOME METRIC PROPERTIES OF AUTOMORPHISMS OF GROUPS

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Abstract. Study of the dynamics of automorphisms of a group is usually focused on their *growth* and/or finite orbits, including fixed points. In this paper, we introduce properties of a different kind; using somewhat informal language, we call them *metric properties*. Two principal characteristics of this kind are called here the "curl" and the "flux"; there seems to be very little correlation between these and the growth of an automorphism, which means they are likely to be an essentially new tool for studying automorphisms.

We also observe that our definitions of the curl and flux are sufficiently general to be applied to mappings of arbitrary metric spaces.

1. Introduction

Let G be a finitely generated group of rank $r \geq 2$ with a set $X = \{x_1, ..., x_r\}$ of generators, and let $|w|$ be the usual lexicographic length of an element $w \in G$ with respect to X.

Let φ be an automorphism (or, more generally, an endomorphism) of G that takes x_i to y_i, $i = 1, ..., r$. The *growth function* of φ with respect to X can be defined as

$$\Gamma_{\varphi,m}(n) = \max_{|w|=m} |\varphi^n(w)|.$$

This function therefore measures, to some extent, how fast the length of elements of G can possibly increase under repeated action of φ.

One can also define a cumulative characteristic, usually called the *growth rate*, or simply *growth*, of φ:

$$\Gamma(\varphi) = \sup_m \limsup_{n\to\infty} \sqrt[n]{\Gamma_{\varphi,m}(n)}.$$

For known properties of growth of automorphisms of a *free group* we refer to [1], [2], and [6]. Very little seems to be known if G is not a free group.

In this paper, we introduce essentially new characteristics of an automorphism. These will tell us how "active" an automorphism is rather than how it "grows".

(1) *Curl function* is defined as

$$Curl_\varphi(n) = |\varphi(B_n) \cap B_n| = \#\{w \in G,\ |w| \leq n,\ |\varphi(w)| \leq n\},$$

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where B_n is the ball of radius n in the Cayley graph of G. This function therefore counts the number of elements left inside the ball of radius n by the automorphism φ.

As with the growth rate, one can define the "curl rate", or simply "curl", of φ as

$$Curl(\varphi) = \limsup_{n\to\infty} \sqrt[n]{\frac{Curl_\varphi(n)}{|B_n|}}.$$

(2) *Flux function* of φ is defined as

$$Flux_\varphi(n) = |B_n \setminus (\varphi(B_n) \cap B_n)| = \#\{w \in G,\ |w| \le n,\ |\varphi(w)| > n\}.$$

This function therefore counts the number of elements taken out of the ball B_n of radius n by the automorphism φ.

Again, one can define the "flux rate", or simply "flux", of φ as follows:

$$Flux(\varphi) = \limsup_{n\to\infty} \sqrt[n]{\frac{Flux_\varphi(n)}{|B_n|}}.$$

We note that all these concepts can be defined for arbitrary *endomorphisms*, not necessarily automorphisms.

It is immediately obvious that:

(i) $0 \le Curl(\varphi),\ Flux(\varphi) \le 1$. It is a very interesting question what values $Curl(\varphi)$ and $Flux(\varphi)$ can actually take. In Section 4, we show that there are gaps on the scale of these values; in particular, $Flux(\varphi)$ cannot take values strictly between 0 and $\frac{1}{4}$ for any injective endomorphism φ of F_r.

(ii) For any n, $Curl_\varphi(n) + Flux_\varphi(n) = |B_n|$, the cardinality of the ball B_n. However, $Curl(\varphi) + Flux(\varphi) \ne 1$ in general; we shall see relevant examples (e.g. Example 3.2) in Section 3.

There are other, less obvious, properties of curl and flux that we have collected in Section 4. Whenever we give a particular property, we use it to compare curl and flux to growth. As it turns out, curl and flux have some useful properties that growth does not have. For instance, we have $Flux(\varphi) = Flux(\varphi^{-1})$ and $Curl(\varphi) = Curl(\varphi^{-1})$ for any automorphism φ (Proposition 4.6); we also have some inequalities for curl and flux functions of composite endomorphisms, including $Flux_{\alpha\beta}(n) \le Flux_\alpha(n) + Flux_\beta(n)$ (Proposition 4.6), etc.

We note at this point that Kaimanovich, Kapovich, and Schupp [3] have independently come up with yet another dynamical characteristic of an automorphism; they call it the *generic stretching factor*. This is a number $\lambda = \lambda(\varphi)$ such that a given automorphism φ "stretches" the length of "almost all" elements of the group approximately by a factor of λ (for more details see our Section 5). This stretching factor appears to be related (although not directly) to our flux. In particular, it is shown in [3] that the flux of any automorphism φ of a *free group* is 1, unless φ is a permutation of the set $X \cup X^{-1}$. Moreover, if φ is not a composition of an inner automorphism and a permutation of the set $X \cup X^{-1}$, then $\lim_{n\to\infty} \frac{Flux_\varphi(n)}{|B_n|} = 1$. Therefore, the flux cannot be used to distinguish automorphisms of a free group.

The situation with the curl however is different. We show, for example, that if $Curl(\varphi) = 1$, then φ is a composition of an inner automorphism and a permutation of the set $X \cup X^{-1}$ (Theorem 5.1 in Section 5). We also show that "stabilizing" an automorphism of a free group (by expanding the free generating set X) may change its curl, but not the growth (see Example 3.4 in Section 3). This is, arguably, an evidence of the curl being a more delicate characteristic of an automorphism than its growth.

To conclude the Introduction, we observe that our definitions of curl and flux are sufficiently general to be applied to mappings of arbitrary metric spaces.

2. Problems

In this section, we list a few open problems that are, in our opinion, important for better understanding the nature of curl and flux. As usual, F_r denotes the free group of rank $r \geq 2$ with a set X of free generators.

Problem 1. (a) What is the maximum (or supremum) of possible $\neq 1$ values of the curl for automorphisms (endomorphisms) of F_r ?

(b) What is the minimum (or infinimum) of possible values of the curl for automorphisms of F_r ?

A good start would be $r = 2$. It is conceivable that the automorphism $\alpha : x \to xy,\ y \to y$ has the maximum possible $\neq 1$ curl among automorphisms of F_2, but we do not have a proof of that. Nor do we have the exact value of $Curl(\alpha)$; according to computer experiments (see Section 5), this value is approximately 0.956.

We also note here that the infinimum of possible values of the curl for *endomorphisms* of F_r is $\frac{1}{2r-1}$, see Proposition 4.1 in Section 4.

Problem 2. What is the minimum (or infinimum) of possible positive values of the flux for endomorphisms of F_r ?

As we have mentioned in the Introduction, $Flux(\varphi)$ cannot take values strictly between 0 and $\frac{1}{4}$ for any injective endomorphism of F_r. If φ is an automorphism of F_r, then $Flux(\varphi) = 0$ or 1 by the result of [3] mentioned in the Introduction. This is however not the case for arbitrary endomorphisms; for example, the endomorphism of F_2 given by $x \to xy,\ y \to 1$ has the flux strictly between 0 and 1 (see Example 3.7 in Section 3).

Problem 3. Are values of flux and curl always algebraic numbers? (Values of growth are.)

Problem 4. Find the *exact* value of $Curl(\varphi)$ for at least one $\varphi \in Aut(F_r)$ with $Curl(\varphi) \neq 1$.

Problem 5. Suppose $Curl(\varphi) = Curl(\psi)$ for some automorphisms φ, ψ of F_r. Is it true that φ is a composition of ψ with a permutation of the set $X \cup X^{-1}$ and an inner automorphism?

The converse is true (see Proposition 4.2 in Section 4). If the answer to Problem 5 is affirmative, this will mean that the curl is indeed a very sharp characteristic of a free group automorphism. We were able to show that if $Curl(\varphi) = 1$, then φ is a composition of an inner automorphism and a permutation of the set $X \cup X^{-1}$ (Theorem 5.1 in Section 5).

The following problem is rather vague, but it appears to be important.

Problem 6. Find tight bounds for $Curl(\alpha\beta)$ in terms of $Curl(\alpha)$, $Curl(\beta)$. More generally, what information about $Curl(\alpha\beta)$ can be extracted from knowing $Curl(\alpha)$ and $Curl(\beta)$?

3. Examples

In this section, we compute curl and flux for some simple automorphisms of F_r, the free group of rank $r \geq 2$ with a set X of free generators.

Example 3.1. Let π be any automorphism that permutes the elements of the set $X \cup X^{-1}$. Then, since π does not change the length of any element, we have $Curl(\pi) = 1$, $Flux(\pi) = 0$. It is also obvious that the growth function of π is identically equal to 1. □

Example 3.2. Let i_g be the conjugation by an element $g \in F_r$. Then $Curl(i_g) = Flux(i_g) = 1$. Indeed, it is sufficient to limit considerations to elements of a sphere S_n because these comprise "most" of the elements of the ball B_n (see [4] for more rigorous estimates supporting this claim). Now suppose g ends with x for some $x \in X \cup X^{-1}$. Then an element $u \in S_n$ gets taken out of B_n by i_g if u does not start with x^{-1}. The number of elements with this property has the same growth function, up to a constant factor, as the total number of elements in S_n does. This yields $Flux(i_g) = 1$.

On the other hand, an element $u \in S_n$ is *not* taken out of B_n by i_g if u starts with g^{-1}. Again, the number of elements with this property has the same growth function, up to a constant factor, as the total number of elements in S_n does. This yields $Curl(i_g) = 1$. □

Example 3.3. Let $r = 2$, and denote the generators of the group F_2 by x and y. Let $\alpha : x \to xy,\ y \to y$. Then the growth function of α is easily seen to be linear in n, whereas both $Curl_\alpha(n)$ and $Flux_\alpha(n)$ are exponential. □

Example 3.4. Again, let $r = 2$, and let i_x be the conjugation by the generator x. Then $Curl(i_x) = Flux(i_x) = 1$. Now extend i_x to the free group F_3 generated by x, y, and z, by fixing the extra generator z. Call this new automorphism $\widehat{i_x}$. Thus, $\widehat{i_x} : x \to x,\ y \to xyx^{-1},\ z \to z$. Then, since $\widehat{i_x}$ is not a composition of an inner automorphism and a permutation of the set $X \cup X^{-1}$, we have $Curl(\widehat{i_x}) < 1$ by Theorem 4.1 in our Section 5. □

Thus, Example 3.4 shows that the curl of an automorphism can change (decrease) under "stabilization". This makes contrast with the growth and reinforces the impression that the curl reflects more delicate properties of automorphisms than the growth does.

In the next example, we show that the curl of an endomorphism can also *increase* under "stabilization".

Example 3.5. Let $r = 2$, and let $\varphi : x \to x^5, y \to y^5$ be an endomorphism of the group F_2. Then, by Proposition 4.1 in Section 4, $Curl(\varphi) = \frac{3^{\frac{1}{5}}}{3}$.

For computational convenience, let us now "stabilize" φ by adding two extra generators, z and t. Thus, $\widehat{\varphi} : x \to x^5, y \to y^5,\ z \to z,\ t \to t$. Then, for any $u = u(z,t)$ of length n, we have $|\varphi(u)| = n$. There are at least 3^n words u like that. Therefore, $Curl(\varphi) \geq \frac{3}{7} > \frac{3^{\frac{1}{5}}}{3}$. $\square$

Example 3.6. Again, let $r = 2$, and let $\varphi = \alpha \cdot \pi_{xy}$, where $\alpha : x \to xy,\ y \to y$, and π_{xy} permutes x and y. Thus, $\varphi : x \to xy,\ y \to x$. Then it is fairly clear that φ has exponential growth (i.e., $\Gamma(\varphi) > 1$), whereas α has linear growth (in particular, $\Gamma(\alpha) = 1$). At the same time, $Flux(\varphi) = Flux(\alpha)$ and $Curl(\varphi) = Curl(\alpha)$ since φ is a composition of α with a length-preserving automorphism. $\square$

The point of this example is to show, again, that the curl and the flux of an automorphism seem to have very little or no correlation with the growth.

We conclude this section with an example of an endomorphism φ of the group F_2 whose flux is strictly between 0 and 1.

Example 3.7. Let $\varphi : x \to xy,\ y \to 1$. Then $0 < Flux(\varphi) < 1$. Indeed, if a word w of length n has $> \frac{n}{2}$ occurrences of x and no occurrences of x^{-1}, then $|\varphi(w)| > n$. The number of words like that is at least $\binom{n}{\frac{n}{2}}$, which is exponential in n. This shows that $0 < Flux(\varphi)$.

To show $Flux(\varphi) < 1$, we observe that for a word w of length n to be taken out of B_n by φ, it should have the exponent sum on x greater than $\frac{n}{2}$ (by the absolute value). This implies that the number of occurrences in w of either x^{-1} or x should be $\geq \frac{3n}{4}$. The set of words like that is exponentially negligible in B_n by [5, Proposition 6.1]. Therefore, $Flux(\varphi) < 1$. $\square$

4. Some properties of curl and flux

In this section, we gather some interesting, in our opinion, properties of curl and flux. Most of these properties are valid for arbitrary *endomorphisms*, not necessarily automorphisms.

Proposition 4.1. **(a)** Let $k \geq 2$, and let $\varphi : x_i \to x_i^k,\ i = 1, ..., r$ be an endomorphism of the group F_r. Then $Curl(\varphi) = \frac{(2r-1)^{\frac{1}{k}}}{2r-1}$.

(b) For any endomorphism ψ of the group F_r, $Curl(\psi) \geq \frac{(2r-1)^{\frac{1}{k}}}{2r-1}$ for some $k \geq 2$. Therefore, the infinimum of possible values of the curl for endomorphisms of F_r is $\frac{1}{2r-1}$.

Proof. **(a)** Note that for any $u \in F_r$, one has $|\varphi(u)| = k|u|$. Therefore, $Curl_\varphi(n)$ is just equal to the number of elements of length $\leq \frac{n}{k}$ in F_r, i.e., to $O((2r-1)^{\frac{n}{k}})$, whence the result.

(b) Let $\psi : x_i \to y_i,\ i = 1, ..., r$, and suppose $|y_i| \leq k$ for some $k \geq 2$. Then $|\psi(u)| \leq k|u|$ for any $u \in F_r$. Therefore, whenever $|u| \leq \frac{n}{k}$, one has $|\psi(u)| \leq n$. The result follows. □

Proposition 4.2. **(a)** Composing any endomorphism φ of F_r with any permutation of the set $X \cup X^{-1}$ does not change either $Flux(\varphi)$ or $Curl(\varphi)$.

(b) Composing any endomorphism φ of F_r with any inner automorphism does not change $Curl(\varphi)$. If φ is injective, then such composing does not change $Flux(\varphi)$ either.

Proof. Part (a) is obvious, so we proceed with part (b). Note that $Curl(\varphi) > 0$ by Proposition 4.1 and $Flux(\varphi) > 0$ by Theorem 4.4 below. Then the argument similar to that in Example 3.2 shows that if we apply φ followed by an inner automorphism, this will not change either $Flux(\varphi)$ or $Curl(\varphi)$.

Suppose now an inner automorphism is applied first, followed by φ. By using inductive argument, we may assume, to simplify the notation, that the inner automorphism is i_x, i.e. conjugation by $x \in X$. Then i_x leaves inside B_n all elements $v \in B_n$ that start with x^{-1}. Suppose now an element $w \in B_n$ starts with some other $y \in X \cup X^{-1}$, i.e., $w = yu$. If this w is left inside B_n by φ, then so is $w^{-1} = u^{-1}y^{-1}$. The number of elements in B_n of the form $u^{-1}y^{-1}$ is the same, up to a constant factor, as the number of elements of the form $x^{-1}u$. Each of these numbers is equal, again up to a constant factor, to the total number of elements in B_n. These two facts show that the curl of the composite endomorphism is the same as the curl of φ. The flux is treated similarly. □

Before we get to the next result, we need a lemma:

Lemma 4.3. Let φ be an endomorphism of F_r such that, for some cyclically reduced $v \in F_r$, one has $|\varphi(v)| \geq 2|v|$. Then $Flux(\varphi) \geq \frac{1}{4}$.

Proof. By Example 3.2, we may assume that φ is *not* a conjugation. We are going to fix a particular k and build sufficiently many words $w \in F_r$ of length k whose length is increased by φ. To that effect, we first fill in the leftmost $\geq \frac{k}{2}$ positions with v^s, where $s = [\log_{|v|} \frac{k}{2}] + 1$. Let $m = |v^s| - \frac{k}{2}$; then $0 \leq m \leq |v|$.

Now we designate the rightmost $\frac{k}{4} - \frac{m}{2} - 1$ positions in w as "arbitrary" (call this part w_{right}), and fill in the intermediate $\frac{k}{4} - \frac{m}{2} + 1$ positions as follows:

(i) Among all words in F_r of length $\frac{k}{4} - \frac{m}{2} - 1$ choose one, call it u, such that $|\varphi(u)| \geq |\varphi(g)|$ for any g of length $\frac{k}{4} - \frac{m}{2} - 1$, and place u immediately left of w_{right}. That way, after we apply φ to w, cancellation between $\varphi(w_{right})$ and $\varphi(u)$ cannot possibly go left beyond $\varphi(u)$.

(ii) Fill in the remaining two positions right of the v^s with two letters, call them a and b, in such a way that there is no cancellation between either $\varphi(v^s)$ and $\varphi(ab)$, or between $\varphi(ab)$ and $\varphi(u)$, or between $\varphi(a)$ and $\varphi(b)$ (this is possible since φ is not a conjugation). Then the length of $\varphi(w)$ is greater than k.

Finally, we observe that the number of different w_{right} of length $\frac{k}{4} - \frac{m}{2} - 1$ grows as $r^{\frac{k}{4}}$, up to an exponentially negligible factor. This yields the result. □

Theorem 4.4. Let φ be an injective endomorphism of F_r. Then either

(a) $Flux(\varphi) = 0$, in which case $|\varphi(x)| = 1$ for all $x \in X$,
or
(b) $Flux(\varphi) \geq \frac{1}{4}$.

Proof. If $|\varphi(x)| = 1$ for all $x \in X$, then obviously $Flux(\varphi) = 0$. Let now $\varphi(x_i) = y_i$, $|y_i| \geq 2$ for some i. Consider two cases:

(1) For some i, $|y_i| \geq 2$ and y_i is cyclically reduced. Then $Flux(\varphi) \geq \frac{1}{4}$ by Lemma 4.3 if we let $v = x_i$.

(2) For all i such that $|y_i| \geq 2$, one has y_i not cyclically reduced. Here we have two subcases:

(i) there are k, l, $k \neq l$, such that for some i, j one has $y_i = x_k g_i x_k^{-1}$, $y_j = x_l g_j x_l^{-1}$, and at least one of the y_i, y_j has length ≥ 2. Then, for $u = x_i x_j$, we have $|\varphi(u)| \geq 2|u|$ and u is cyclically reduced. Then, by Lemma 4.3, we have $Flux(\varphi) \geq \frac{1}{4}$.

(ii) every y_i with $|y_i| \geq 2$ is of the form xg_ix^{-1} for some fixed $x \in X \cup X^{-1}$. Suppose, for some j, $\varphi(x_j) = x_k \neq x$. Then, for $u = x_i x_k$, we have $|\varphi(u)| \geq 2|u|$ and u is cyclically reduced. Then, by Lemma 4.3, we have $Flux(\varphi) \geq \frac{1}{4}$. The remaining case is where *every* y_i is of the form xg_ix^{-1}. If, for some i, $|g_i| \geq 2$, then the argument from the proof of Lemma 4.3 will still work after obvious minor adjustments. If $|g_i| = 1$ for every i, then φ is a composition of a permutation with the conjugation by x, whence $Flux(\varphi) = 1$. □

Proposition 4.5. For any automorphism φ of any group G, $Flux(\varphi) = Flux(\varphi^{-1})$ and $Curl(\varphi) = Curl(\varphi^{-1})$. Moreover, for any $n \geq 1$, $Flux_{\varphi^{-1}}(n) = Flux_{\varphi}(n)$, and $Curl_{\varphi^{-1}}(n) = Curl_{\varphi}(n)$.

Proof. Let A be the set of elements of B_n taken out of B_n by φ, and B the set of elements of B_n left by φ inside the ball. Furthermore, let C be the set of elements outside of B_n taken by φ inside B_n, A' the set of elements of B_n taken out of B_n by φ^{-1}, and B' the set of elements of B_n left by φ^{-1} inside the ball.

Then, since φ is *onto*, we must have $|C| = |A|$. At the same time, we clearly have $|C| = |A'|$, hence $|A'| = |A|$. This implies $Flux_{\varphi^{-1}}(n) = Flux_{\varphi}(n)$.

Now since $A \cup B = A' \cup B' = B_n$ and $A \cap B = A' \cap B' = \emptyset$, we have $|B'| = |B|$, whence $Curl_{\varphi}(n) = Curl_{\varphi^{-1}}(n)$. □

Proposition 4.6. For any automorphisms α and β of any group G and for any $n \geq 1$, one has:

(a) $Curl_{\alpha\beta}(n) \leq Curl_{\beta}(n) + Flux_{\alpha}(n)$.

(b) $Flux_{\alpha\beta}(n) \leq Flux_{\alpha}(n) + Flux_{\beta}(n)$.

(c) $Curl_{\alpha\beta}(n) \geq Curl_{\beta}(n) - Flux_{\alpha}(n)$.

(d) $Flux_{\alpha\beta}(n) \geq Flux_{\beta}(n) - Flux_{\alpha}(n)$ and $Flux_{\alpha\beta}(n) \geq Flux_{\alpha}(n) - Flux_{\beta}(n)$.

(e) $Flux_{\alpha\beta}(n) \geq Curl_{\alpha}(n) - Curl_{\beta}(n)$.

Inequalities (a) and (b) are actually valid for arbitrary endomorphisms.

Proof. First of all, we note that when we write $\alpha\beta$, we assume that α is applied first.

(a) Elements left inside the ball B_n by the automorphism $\alpha\beta$ are among those left inside B_n by β or among those first taken by α outside B_n, and then taken back inside by β. The quantity of the former is bounded by $Curl_\beta(n)$, and the quantity of the latter by $Flux_\alpha(n)$. This completes the proof of part (a).

(b) Argument similar to the one in (a) establishes this inequality.

(c) In (a), plug in α^{-1} for α and $\alpha\beta$ for β. Then observe that $Flux_\alpha(n) = Flux_{\alpha^{-1}}(n)$ by Proposition 4.5.

(d) Re-write (b) as $Flux_\alpha(n) \geq Flux_\beta(n) - Flux_{\alpha\beta}(n)$. Now plug in $\alpha\beta$ for α and β^{-1} for β to get $Flux_{\alpha\beta}(n) \geq Flux_{\beta^{-1}}(n) - Flux_\alpha(n)$. Since, by Proposition 4.5, $Flux_{\beta^{-1}}(n) = Flux_\beta(n)$, this yields the first inequality.

For the second inequality, plug in $\alpha\beta$ for α and β^{-1} for β in (b). Then we get $Flux_\alpha(n) \leq Flux_{\alpha\beta}(n) + Flux_\beta^{-1}(n)$. Since $Flux_{\beta^{-1}}(n) = Flux_\beta(n)$ by Proposition 4.5, this yields the result.

(e) In (a), plug in $\alpha\beta$ for α and β^{-1} for β. Then observe that $Curl_\beta(n) = Curl_{\beta^{-1}}(n)$ by Proposition 4.5. □

5. Evaluating the curl

Computing the exact value of $Curl(\varphi)$ is a difficult problem for most automorphisms φ of a free group, so the best one can hope for (at least for now) is to somehow estimate that value. In this section, we are able to give the affirmative answer to Problem 5 from Section 2 in the special case where ψ is the identity automorphism.

To fully appreciate Theorem 5.1 below, the reader should bear in mind that, according to computer experiments (see the tables in the end of this section), for the automorphism $\varphi : x \to xy,\ y \to y$ of F_2, $Curl(\varphi)$ is approximately 0.956.

Theorem 5.1. Let α be an automorphism of the group F_r which is not a composition of an inner automorphism and a permutation of the set $X \cup X^{-1}$. Then $Curl(\alpha) < 1$. Moreover, $Curl(\alpha)$ is bounded away from 1, i.e., there is a constant $c = c(r) < 1$, independent of α, such that $Curl(\alpha) < c$.

Proof. To simplify the language, let us call an automorphism *simple* if it is a composition of an inner automorphism and a permutation of the set $X \cup X^{-1}$.

Denote by $M_n(\alpha)$ the set $\{w \in F_r,\ |w| \leq n,\ |\alpha(w)| \leq n\}$. Recall that the cardinality of this set is what we call the curl function $Curl_\alpha(n)$ of α.

Let $\lambda > 1$. Clearly,

$$M_n(\alpha) \ \subseteq\ B_{\frac{n}{\lambda}} \cup \{u \in F_r,\ \frac{n}{\lambda} < |u| \leq n,\ |\alpha(u)| \leq \lambda |u|\}.$$

The first set in the union on the right, the ball of radius $\frac{n}{\lambda}$, is *asymptotically exponentially negligible compared to* B_n (or just *asymptotically exponentially negligible*, to simplify the language), which means

$$\lim_{n\to\infty} \sqrt[n]{\frac{|B_{\frac{n}{\lambda}}|}{|B_n|}} < 1.$$

By [3, Theorem 6.8], if $\lambda < 1 + \frac{2r-3}{2r^2-r}$, then, since α is not simple, the second set in the union above, i.e., the set

$$S_{\lambda,\alpha}(n) = \{u \in F_r,\ \frac{n}{\lambda} < |u| \le n,\ |\alpha(u)| \le \lambda |u|\},$$

must be asymptotically exponentially negligible, too.

Since the union of two asymptotically exponentially negligible sets is itself asymptotically exponentially negligible, this implies that the set M_n is asymptotically exponentially negligible, hence $Curl(\alpha) < 1$.

To prove the last claim in the statement of Theorem 5.1, we note that, for a fixed λ such that $1 < \lambda < 1 + \frac{2r-3}{2r^2-r}$, both the limits $\lim_{n\to\infty} \sqrt[n]{\frac{|B_{\frac{n}{\lambda}}|}{|B_n|}}$ and $\lim_{n\to\infty} \sqrt[n]{\frac{|S_{\lambda,\alpha}(n)|}{|B_n|}}$ are bounded away from 1 by a constant $c = c(r) < 1$, independent of α. For the former limit, this is obvious. For the latter limit, this follows from the argument in the beginning of the proof of [3, Theorem 6.8]. □

In conclusion, we present the results of computer experiments on evaluating flux and curl of several automorphism. In the tables below, we give values of the *curl ratio* $\frac{Curl_\varphi(n)}{|B_n|}$ and the *flux ratio* $\frac{Flux_\varphi(n)}{|B_n|}$ along with the *curl root* $\sqrt[n]{\frac{Curl_\varphi(n)}{|B_n|}}$ and the *flux root* $\sqrt[n]{\frac{Flux_\varphi(n)}{|B_n|}}$.

We start with the "simplest non-simple" automorphism of F_2.

$$\begin{cases} x \to xy \\ y \to y \end{cases}$$

n	CURL_RATIO	CURL_ROOT	FLUX_RATIO	FLUX_ROOT
10	0.331634	0.895501	0.668366	0.960509
20	0.181176	0.918132	0.818824	0.990055
50	0.0372579	0.93632	0.962742	0.999241
100	0.0033803	0.94469	0.99662	0.999966
200	3.55979e-05	0.950073	0.999964	1
300	4.20992e-07	0.952243	1	1
400	5.23913e-09	0.95345	1	1
500	6.71114e-11	0.954231	1	1
600	8.75867e-13	0.954782	1	1
700	1.15812e-14	0.955193	1	1
800	1.54618e-16	0.955513	1	1
900	2.04046e-18	0.95575	1	1
1000	2.78188e-20	0.95597	1	1

In the next table, we treat the "stabilization" of the previous automorphism. We see that the curl of the "stabilization" is apparently smaller.

$$\begin{cases} x \to xy \\ y \to y \\ z \to z \end{cases}$$

n	CURL_RATIO	CURL_ROOT	FLUX_RATIO	FLUX_ROOT
10	0.220658	0.85975	0.779342	0.975378
20	0.0832884	0.883139	0.916712	0.995661
50	0.00616004	0.903216	0.99384	0.999876
100	0.000106955	0.912624	0.999893	0.999999
200	4.26719e-08	0.918651	1	1
300	1.93205e-11	0.921057	1	1
400	9.23441e-15	0.922388	1	1
500	4.52035e-18	0.923231	1	1

In the next table, we treat the square of the first automorphism. We see that the curl of the square is apparently smaller than that of the automorphism itself.

$$\begin{cases} x \to xy^2 \\ y \to y \end{cases}$$

n	CURL_RATIO	CURL_ROOT	FLUX_RATIO	FLUX_ROOT
10	0.143331	0.823444	0.856670	0.984649
20	0.0408009	0.852184	0.959199	0.997919
50	0.00133009	0.875947	0.99867	0.999973
100	5.98358e-06	0.886686	0.999994	1
200	1.61895e-10	0.8934	1	1
300	4.98636e-15	0.896037	1	1
400	1.36942e-19	0.897101	1	1

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