Factor algebras of free algebras: on a problem of G. Bergman

Vladimir Shpilrain

and

Jie-Tai Yu*

Abstract. Let $A_n = K\langle x_1, ..., x_n\rangle$ be a free associative algebra over a field K. In this paper, we give examples of elements $u \in A_n$, $n \geq 3$, such that the factor algebra of A_n over the ideal generated by u is isomorphic to A_{n-1}, and yet u is not a primitive element of A_n (i.e., it cannot be taken to x_1 by an automorphism of A_n). If the characteristic of the ground field K is 0, this yields a negative answer to a question of G. Bergman. On the other hand, by a result of Drensky and Yu, there is no such example for $n = 2$.

We note that a similar question for *polynomial algebras* is a major open problem in affine algebraic geometry, known as the Embedding conjecture of Abhyankar and Sathaye.

1. Introduction

Let $A_n = K\langle x_1, ..., x_n\rangle$ be a free associative algebra over a field K. For $u \in A_n$, let $\langle u\rangle$ denote the ideal of A_n generated by u. In this paper, we give examples of elements $u \in A_n$, $n \geq 3$, such that the factor algebra of A_n over the ideal $\langle u\rangle$ is isomorphic to A_{n-1}, and yet u is not a primitive element of A_n (i.e., it cannot be taken to x_1 by an automorphism of A_n).

Theorem 1.1. Let $u = x_1 - (x_1^2 + x_2x_3)x_3$. Then, for any $n \geq 3$, the factor algebra $A_n/\langle u\rangle$ is isomorphic to A_{n-1}, but u is not a primitive element of A_n.

This, combined with Makar-Limanov's "Freiheitssatz" [7], gives a negative answer to a question of G. Bergman [3, p. 336, Problem 6]:

Corollary 1.2. Let K be a field of characteristic 0. Let $u = x_1 - (x_1^2 + x_2x_3)x_3$. Then, for any $n \geq 3$, the factor algebra $A_n/\langle u\rangle$ is isomorphic to A_{n-1}, but no automorphism of A_n takes the ideal $\langle u\rangle$ to $\langle x_1\rangle$.

We note that $u = x_1 - (x_1^2 + x_2x_3)x_3$ is just one of the many elements providing a negative answer to Bergman's question. Other examples include $u_{k,m} = x_1 - (x_1^k + x_2x_3^m)x_3^m$ for any $k \geq 2,\ m \geq 1$.

If $n = 2$ and the ground field has characteristic 0, the isomorphism of $A_2/\langle u\rangle$ to A_1 does imply that u is a primitive element of A_2. This was proved by Drensky and Yu

2000 Mathematics Subject Classification: Primary 16S10, 16W20; Secondary 14A05, 13B25.

*) Partially supported by RGC Grant Project HKU 7134/00P.

in [4], but to make the exposition here complete, we give a (slightly different) proof of this fact in Section 3.

We also mention here a related question for *polynomial algebras* $P_n = K[x_1, ..., x_n]$, known as the Embedding conjecture of Abhyankar and Sathaye [1], [9]:

The Embedding conjecture. Let $u \in P_n$, $n \geq 2$. If the factor algebra $P_n/\langle u \rangle$ is isomorphic to P_{n-1}, then u can be taken to x_1 by an automorphism of P_n.

This conjecture was settled in the affirmative for $n = 2$ (if the ground field has characteristic 0) by Abhyankar and Moh [1], but for $n \geq 3$, it remains a major open problem. We also note that if the ground field has a positive characteristic p, then there is a counterexample to the Embedding conjecture even for $n = 2$ (and therefore, for any bigger n as well): $u(x_1, x_2) = x_1 + x_1^{p(p+1)} + x_2^{p^2}$.

Finally, we formulate two open problems relevant to the subject of this paper.

Problem 1. Is there a primitive element $v \in K\langle x, y, z \rangle$ with the same abelianization as $x - (x^2 + yz)z$?

If there is not, then Nagata's automorphism [8] is not tame because the abelianization of $x - (x^2 + yz)z$ is a component of Nagata's automorphism of the polynomial algebra $K[x, y, z]$. Obviously, every tame automorphism of $K[x, y, z]$ can be lifted to a tame automorphism of $K\langle x, y, z \rangle$.

Problem 2. Let $u \in A_n$, and suppose that $\varphi(u) = x_1$ for some injective endomorphism φ of A_n. Is it true that u is a primitive element of A_n ?

This problem is open for $n = 2$ (see [4] for discussion) and for $n = 3$. For $n \geq 4$, we give a counterexample below (Example 1). We note that a similar problem for polynomial algebras has a positive solution for $n = 2$ [2] and a negative solution for $n \geq 3$ (Makar-Limanov, personal communication). We reproduce Makar-Limanov's example below (Example 2) in the hope that it might give some insight into Problem 2 for $n = 3$.

Example 1. Let $\phi : x \to x + yt,\ y \to tz + 1,\ z \to z,\ t \to -xt$. This is an injective mapping from $K\langle x, y, z, t \rangle$ into itself. Let $u = u(x, y, z, t) = xy + tz$. Then $\phi(u) = x + yt^2z + yt$. Thus, a composition of ϕ with an automorphism takes the element u to x. On the other hand, there is no automorphism of $K\langle x, y, z, t \rangle$ that would take u to x because u has zero linear part.

Example 2. Let $\phi : x \to x,\ y \to 1 + 2yz + xy^2,\ z \to z$. This is obviously an injective mapping from $K[x, y, z]$ into itself. Let $p = p(x, y, z) = xy + z^2$. Then $\phi(p) = x + (z + xy)^2$. Thus, a composition of ϕ with an automorphism takes the polynomial p to x. On the other hand, there is obviously no automorphism of $K[x, y, z]$ that would take p to x.

2. Preliminaries

We are going to need some background on *Fox derivatives* in free ideal rings in general and in free associative algebras in particular (see [3] or [5] for details, although in the former monograph the terminology is different).

The *augmentation ideal* Δ of the algebra A_n is the kernel of the map $x_i \to 0,\ i = 1, ..., n$. It is a free left (and right) A_n-module with a free basis $\{x_1, ..., x_n\}$, so that for any $u \in \Delta$, there is a unique expression of the form $u = x_1 \cdot d_1(u) + ... + x_n \cdot d_n(u)$. The elements $d_j(u)$ are called (right) Fox derivatives.

One can extend these derivations linearly to the whole A_n by setting $d_i(1) = 0$.

If $\varphi \in Aut(A_n)$ takes x_i to $u_i,\ i = 1, ..., n$, then $J_\varphi = (d_i(u_j))_{1 \le i,j \le n}$ is the Jacobian matrix of φ.

There is a "chain rule" for Fox derivatives (see e.g. [5]) that implies the following product rule for the Jacobian matrices (it is, in fact, the same in the commutative and the non-commutative situation):

$$J_{\varphi\psi} = J_\varphi \cdot \varphi(J_\psi).$$

When we write a product $\varphi\psi$, that means ψ is applied first. When we write $\varphi(J_\psi)$, that means φ is applied to each entry of J_ψ.

The proposition below follows immediately from the product rule for the Jacobian matrices:

Proposition 2.1. If $u \in A_n$ is primitive, then the Fox gradient $(d_1(u)), ..., d_n(u))$ is left unimodular, i.e., the Fox derivatives of u generate the whole A_n as a left ideal.

We note that the converse of Proposition 2.1 does not hold; a simple example would be $u = x_1 + x_2 x_1$. The Fox gradient of this element is $(1,\ x_1)$, which is left unimodular. However, u is not primitive since it can be factored as $(1 + x_2)x_1$.

3. Proofs

Proof of Theorem 1.1. The fact that u is not primitive follows from Proposition 2.1 since the Fox gradient of u is $(1 - x_1 x_3,\ \ - x_3^2,\ 0)$, which is not left unimodular.

Now we are going to show that the factor algebra $A_n/\langle u \rangle$ is isomorphic to A_{n-1}. It will be technically more convenient to write factor algebras as "algebras with relations", i.e., for example, instead of $A_n/\langle u \rangle$ we shall write $\langle x_1, ..., x_n \mid u = 0 \rangle$. Following is the chain of isomorphism-preserving transformations (similar to Tietze transformations in group theory – see e.g. [6]) that establishes the isomorphism between
$\langle x, y, z \mid x = (x^2 + yz)z \rangle$ and $\langle x, y, z \mid x = 0 \rangle$.

$\langle x, y, z \mid x = (x^2 + yz)z \rangle \ \cong\ \langle x, y, z, t \mid x = (x^2 + yz)z,\ t = x^2 \rangle$.

Here we have added an extra variable t which is expressed in terms of other variables. This transformation obviously induces an isomorphism of factor algebras.

$\langle x, y, z, t \mid x = (x^2 + yz)z,\ t = x^2 \rangle \ \cong\ \langle x, y, z, t \mid x = (t + yz)z,\ t = x^2 \rangle$.

Now x is expressed in terms of other variables, so we can get rid of x:
$\langle x, y, z, t \mid x = (t + yz)z,\ t = x^2 \rangle \ \cong\ \langle y, z, t \mid t = ((t + yz)z)^2 \rangle$.

Now we just rename the variables, and then apply the automorphism $x \to x - yz,\ y \to y,\ z \to z$:
$\langle y, z, t \mid t = ((t + yz)z)^2 \rangle \ \cong\ \langle x, y, z \mid x = ((x + yz)z)^2 \rangle \ =$
$\langle x, y, z \mid x = (x + yz)z(x + yz)z \rangle \ \cong\ \langle x, y, z \mid x - yz = xzxz \rangle$.

The rest is straightforward:
$\langle x, y, z \mid x - yz = xzxz \rangle \ \cong\ \langle x, y, z \mid x = (xzx + y)z \rangle \ \cong\ \langle x, y, z \mid x = yz \rangle \ \cong$
$\langle x, y, z \mid x = 0 \rangle$. $\square$

Proof of Corollary 1.2. Let $u = x_1 - (x_1^2 + x_2x_3)x_3 \in K\langle x_1, ..., x_n \rangle$, and suppose there is a primitive element $p \in \langle u \rangle$. Apply an automorphism ϕ that takes p to x. Then $x \in \langle \phi(u) \rangle$. But $\phi(u)$ depends on at least two variables because it is not primitive and irreducible; therefore, by the Freiheitssatz [7], every element of $\langle u \rangle$ depends on at least two variables, too. This contradiction completes the proof. $\square$

Proposition 3.1. (cf. [4]) Let K be a field of characteristic 0, and let $u \in K\langle x, y \rangle$. If the factor algebra $K\langle x, y \rangle / \langle u \rangle$ is isomorphic to $K\langle x \rangle$, then u is a primitive element of $K\langle x, y \rangle$.

Proof. Let $u \in K\langle x, y \rangle$, and let $K\langle x, y \rangle / \langle u \rangle$ be isomorphic to $K\langle x, y \rangle / \langle x \rangle$. Let $u^a \in K[x, y]$ be the natural abelianization of u. Then $K[x, y]/\langle u^a \rangle$ is isomorphic to $K[x, y]/\langle x \rangle$. By Abhyankar-Moh's theorem [1], u^a is a *coordinate* in $K[x, y]$, i.e., it can be taken to x by an automorphism of $K[x, y]$.

Since there is an isomorphism φ between $K\langle x, y \rangle / \langle u \rangle$ and $K\langle x, y \rangle / \langle x \rangle$, we see that every element of the commutator ideal of $K\langle x, y \rangle$ belongs to the ideal $\langle x \rangle$; otherwise, φ would not be one-to-one because $\varphi(xy - yx) \in \langle x \rangle$ for any φ. Therefore, the ideal $\langle u \rangle$ contains all elements whose abelianization is u^a. One of them must be primitive; then, as in the proof of Corollary 1.2, we conclude that u must be primitive by the Freiheitssatz [7]. $\square$

Acknowledgements

We are grateful to Warren Dicks for helpful discussions and for pointing out G. Bergman's question to us.

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Department of Mathematics, The City College of New York, New York, NY 10031
e-mail address: shpil@groups.sci.ccny.cuny.edu
http://www.sci.ccny.cuny.edu/~shpil/

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
e-mail address: yujt@hkusua.hku.hk
http://hkumath.hku.hk/~jtyu