Non-commutative determinants and automorphisms of groups

Vladimir Shpilrain

Abstract

We show that for any invertible matrix over an arbitrary group algebra, it is possible to define a series of invariants which have most of the properties of the usual commutative determinants. We also give one of the applications of these invariants — to detecting Nielsen inequivalent generating systems of various groups.

1 Introduction

Let $F = F_n$ be the free group of a finite rank $n \geqslant 2$ with a fixed set $\{x_i\}$, $1 \leqslant i \leqslant n$, of free generators, and let R be a normal subgroup of F. Consider the group $G = F/R$; two generating systems of this group of the same cardinality n are called *Nielsen equivalent* if one of them can be obtained from another by applying a sequence of Nielsen transformations (see e.g. [16]). The question of whether or nor any two minimal generating systems of a given group are Nielsen equivalent has important connections with low-dimensional topology — see e.g. [15]. This question is equivalent to the following one: is every generating system of a group $G = F/R$ Nielsen equivalent to the "canonical" one $\{x_1R, \ldots, x_nR\}$? Generating systems with this property are called *tame*; one can also say that such a system can be *lifted* to a generating system of F. *Tame automorphisms* of a group G are defined in a similar fashion.

The questions of lifting automorphisms / generating systems are naturally related to the problem of finding appropriate necessary and / or sufficient condition(s) for an endomorphism of the group F to be an automorphism.

Nielsen [17] (see also [16]) has described the group $\operatorname{Aut} F$ in terms of generators and relations and has also given an algorithm for deciding if n given elements of the group F generate this group. However Nielsen's procedure is not suitable for our purposes since the elements under consideration are not given explicitly.

Birman [6] has given a matrix characterization of automorphisms among arbitrary endomorphisms as follows. Define the matrix $J_\phi = (d_j(y_i))_{1 \leqslant i,j \leqslant n}$ (the "Jacobian matrix" of ϕ), where d_j denotes partial Fox derivation with respect to x_j in the free group ring $\mathbb{Z}F$ (see [12]). Then ϕ is an automorphism if and only if the matrix J_ϕ is invertible over $\mathbb{Z}F$. This result has been later generalized by Krasnikov [14]. He has proved the following: let R be an arbitrary normal subgroup of F, and let $\sigma_R\colon F \to F/R$ be the natural homomorphism; it can be linearly extended to a homomorphism of group rings $\sigma_R\colon \mathbb{Z}F \to \mathbb{Z}(F/R)$. Then elements $y_1, \ldots, y_n$ generate the group F modulo R' if and only if the matrix $(\sigma_R(d_j(y_i)))_{1\leqslant i,j\leqslant n}$ is invertible over the ring $\mathbb{Z}(F/R)$.

It is this result of Birman that has been used as the initial point in obtaining two powerful necessary conditions of tameness known by now.

The first approach comes from [8]; it gives a necessary condition for a matrix of a special form with the entries from the integral group ring $\mathbb{Z}F$ to be invertible. This condition does not formally use the fact that a matrix under consideration is the Jacobian matrix of an automorphism. On the other hand, the second necessary condition which is due to Lustig and Moriah [15], is explicitly based on this fact: they consider the image of the Jacobian matrix in an appropriate ring $M_n(K)$ with K a commutative ring with 1, evaluate the determinant of this image and then observe that this determinant should be equal to a *trivial unit* of the ring K, i.e., to an element of the (multiplicative) group generated by the images of $\pm x_1, \ldots, \pm x_n$.

The method we are going to present in this paper has advantages of both just mentioned. We consider a matrix over $\mathbb{Z}F$, then evaluate something that serves as a non-commutative analog of the determinant of this matrix and observe that this non-commutative determinant must satisfy a condition similar to that of [8]. This is described in our main Section 3. Furthermore, we show that our method can be applied to matrices over arbitrary integral group rings as well as to matrices over group algebras $\mathbf{K}G$, $\mathbf{K}$ a field. In fact, we give a procedure for detecting non-invertibility of an arbitrary square matrix over an arbitrary group algebra (its effectivness though depends on what we know about group algebra itself, so it is not an algorithm in the usual sense).

In the concluding Section 4, we give some applications of this technique to detecting non-tame automorphisms / generating systems.

First we present a non-tame automorphism of the free 2-metabelian group $F_3/(F_3')^2$ of rank 3 (Theorem 4.1) contributing toward a solution of Problem 4 from [4]; now this problem remains unsettled only for free p-metabelian groups $F_3/(F_3')^pF''$ of rank 3 when $p > 2$ is a prime. The case when p is not a prime has been recently settled by Papistas.

Also, upon combining our method with the above cited Krasnikov's result, we show that in many cases, if an automorphism of a group F/R cannot be lifted to an automorphism of F, then in fact it cannot be lifted to an automorphism of the group F/R'. In particular, we show that there are automorphisms of the free metabelian group of rank 3, which cannot be lifted to automorphisms of the free solvable group of rank 3 and derived length 3 (Theorem 4.4). This completes the answer to Problem 5.49 (b) from [13] due to Remeslennikov (Corollary 4.5); other cases of this problem are covered by results of [19].

2 Preliminaries

Let $\mathbf{K}F$ be the group algebra of the group F over a commutative domain $\mathbf{K}$, and Δ its augmentation ideal, that is, the kernel of the natural homomorphism $\sigma\colon \mathbf{K}F \to \mathbf{K}$. When $R \neq F$ is a normal subgroup of F, we denote by Δ_R the ideal of $\mathbf{K}F$ generated by all elements of the form $(r-1)$, $r \in R$. It is the kernel of the natural homomorphism $\sigma_R\colon \mathbf{K}F \to \mathbf{K}(F/R)$.

The ideal Δ is a free left $\mathbf{K}F$-module with a free basis $\{(x_i - 1)\}$, $1 \leqslant i \leqslant n$, and Fox derivations d_i are projections to the corresponding free cyclic direct summands. Thus any element $u \in \Delta$ can be uniquely written in the form $u = \sum_i d_i(u)(x_i - 1)$.

One can extend these derivations linearly to the whole $\mathbf{K}F$ upon defining $d_i(1) = 0$.

The following lemma is an immediate consequence of the definitions.

Lemma 2.1. *Let J be an arbitrary right ideal of $\mathbf{K}F$ and let $u \in \Delta$. Then $u \in J\Delta$ if and only if $d_i(u) \in J$ for each i, $1 \leqslant i \leqslant n$.*

Proof of the next lemma can be found in [12].

Lemma 2.2. *Let R be a normal subgroup of F, and let $y \in F$. Then:*

(i) *In the ring $\mathbb{Z}F$, $y - 1 \in \Delta_R\Delta$ if and only if $y \in R'$.*

(ii) *In the ring $\mathbb{Z}_mF$, $m \geqslant 2$, $y - 1 \in \Delta_R\Delta$ if and only if $y \in R^mR'$.*

(iii) *In the ring $\mathbb{Z}F$, $y - 1 \in \Delta^c$ if and only if $y \in \gamma_c(F)$, $c \geqslant 1$.*

(iv) (*see* [18]). *In the ring $\mathbb{Z}_pF$, p a prime, $y - 1 \in \Delta^c$ if and only if $y \in \gamma_c^{(p)}(F)$, $c \geqslant 1$, where $\gamma_c^{(p)}(F)$ denotes the c-th term of the* Jennings series *defined inductively: $\gamma_1^{(p)}(F) = F$; $\gamma_{c+1}^{(p)}(F) = [\gamma_c^{(p)}(F), F] \cdot (\gamma_m^{(p)}(F))^p$, where m is the minimal integer such that $mp \geqslant c$. (In particular, $\gamma_2^{(p)}(F) = F'F^p$.)*

We also need the "chain rule" for Fox derivations (see [12]):

Lemma 2.3. *Let $v = v(x_1, \ldots, x_m)$ be an element of $\mathbf{K}F$, and $y_1, \ldots, y_m$ — elements of F. Denote $u_k(x_1, \ldots, x_m) = d_k(v)$. Then:*

$$d_j(v(y_1, \ldots, y_m)) = \sum_{1 \leqslant k \leqslant m} u_k(y_1, \ldots, y_m) d_j(y_k).$$

Lemma 2.3 implies the following product rule for the Jacobian matrices which looks exactly the same as in the "usual" situation of analytic functions and Leibnitz derivations: if ϕ and ψ are two endomorphisms of F, then

$$J_{\phi\psi} = \psi(J_\phi) \cdot J_\psi, \tag{1}$$

where by $\phi\psi$ we mean $\phi(\psi)$.

We also note that Fox derivatives of higher weight can be defined in the obvious way; we denote $d_{ij}(u) = d_i(d_j(u))$.

Now we need one remark about Fox derivatives in *integral* group rings. Whereas an arbitrary element of the ring $\mathbb{Z}F$ can be a Fox derivative of some other element of $\mathbb{Z}F$, Fox derivatives of *group elements* satisfy several conditions which have been given in [10] in the form of relations between the augmentations of derivatives of different weights. We are not going to give all of them; the following is sufficient for our purposes:

Lemma 2.4. *Let $g \in F$. Then:*

(i) $(\sigma(d_i(g)))^2 = \sigma(d_i(g)) + 2\sigma(d_{ii}(g))$.

(ii) *If $g \in \gamma_c(F)$, $c \geqslant 2$, and $\{i_1, \ldots, i_c\}$ is a set of indices with $i_1 \neq i_j$, $j \geqslant 2$, then*

$$\sigma(d_{i_1 i_2 \ldots i_c}(g)) + \sigma(d_{i_2 i_1 \ldots i_c}(g)) + \sigma(d_{i_2 i_3 \ldots i_c i_1}(g)) = 0.$$

This equality holds also in group rings $\mathbb{Z}_p F$, p a prime, on replacing $\gamma_c(F)$ by $\gamma_c^{(p)}(F)$.

We also need the following fact:

Lemma 2.5. *Let $g \in \gamma_c(F)$, $c \geqslant 2$. Then $g - 1 \in (\Delta, \Delta) + \Delta^{c+1}$, where (Δ, Δ) denotes the additive subgroup of $\mathbb{Z}F$ generated by all elements of the form $xy - yx$; $x, y \in \Delta$.*

Proof. We have

$$xy - 1 = (x-1)y + y - 1;$$
$$[g, h] - 1 = g^{-1}h^{-1}((g-1)(h-1) - (h-1)(g-1)),$$

where $g, h \in F$, and $[g, h] = g^{-1}h^{-1}gh$. The proof now follows easily by induction on c and on the number of commutators of weight c in a decomposition of g modulo $\gamma_{c+1}(F)$. □

3 Non-commutative determinants

Our construction of non-commutative determinants can be considered a generalization of the corresponding construction due to Dieudonné (see [1] for a very detailed exposition), which applied to matrices over skew fields. The argument in [1] applies to a more general situation as follows.

Let S be an associative ring with 1 and with the following property:

(*) every invertible square matrix with the entries from S (i.e., a matrix from the group $\mathrm{GL}_k(S)$ for some $k \geqslant 1$) has at least one invertible element in every row and in every column.

Among the rings with this property are, for example, all local rings, i.e., rings whose non-invertible elements form an ideal.

Over such a ring, any invertible square matrix M can be written in a special form $M = E \cdot D(\mu)$, where E is a product of *elementary matrices* (i.e., matrices possibly different from the identity matrix by a single element outside the diagonal), and $D(\mu)$ is the matrix that is obtained from the diagonal matrix by replacing the "1" in the right lower corner with $\mu \in S$, $\mu \neq 0$. Take now the group S^* of all invertible elements of the ring S, and the quotient group $\bar{S}$ of S^* by its commutator subgroup. Then the image $\bar{\mu}$ of μ in $\bar{S}$ is defined uniquely and is called the determinant of the matrix M in the sense of Dieudonné. It has some usual properties of the determinant; in particular, the determinant of the product of two matrices from $\mathrm{GL}_k(S)$ is equal to the product of their determinants.

Now given a matrix $M \in \mathrm{GL}_n(\mathbf{K}G)$ over an arbitrary group algebra, we consider a ring $S_m = \mathbf{K}G/\Delta^m$, $m \geqslant 2$, and come up with the determinant $\delta_m(M)$ in accordance with the construction described above using the fact that in the ring S_m, every element with non-zero augmentation is invertible.

More specially, given an element $u \in \mathbf{K}G$ of the form $\alpha - w$, $0 \neq \alpha \in \mathbf{K}$, $w \in \Delta$, one has modulo Δ^m: $u^{-1} = \alpha^{-1} + \alpha^{-2}w + \cdots + \alpha^{-m}w^{m-1}$. Hence the commutator subgroup of the group S_m^* is generated as a group (actually as a semigroup) modulo Δ^m by elements of the form

$$(1 - v)(1 - w)(1 + v + \cdots + v^{m-1})(1 + w + \cdots + w^{m-1}) \tag{2}$$

with $v, w \in \Delta$. Denote by P_m the (multiplicative) subsemigroup of $\mathbf{K}G$ generated by all elements of the form (2). We have:

Proposition 3.1. *Let $M \in \mathrm{GL}_n(\mathbf{K}G)$ be a product of elementary and diagonal matrices. Let $\det_m(M)$ be an arbitrary preimage (in $\mathbf{K}G$) of $\delta_m(M)$. Then for an arbitrary $m \geqslant 2$, one has*

$$\det_m(M) = u_m p_m + w_m \tag{3}$$

for some invertible element $u_m \in \mathbf{K}G$; $p_m \in P_m$; $w_m \in \Delta^m$.

In the case when $G = F$ is a free group, we therefore have:

$$\det_m(M) = \pm g_m p_m + w_m, \tag{4}$$

$g_m \in F$; $p_m \in P_m$; $w_m \in \Delta^m$.

We also note that by [5, IV.5.17], every matrix from $\mathrm{GL}_n(\mathbf{K}F)$, $n \geqslant 2$, $\mathbf{K}$ a field, is a product of elementary and diagonal matrices because $\mathbf{K}F$ is a free ideal ring, so that (4) gives necessary condition(s) for a matrix over $\mathbf{K}F$ to be invertible.

In the case of integral group algebra $\mathbb{Z}G$, the rings $\mathbb{Z}G/\Delta^m$ don't satisfy the property (*), but still we can write any invertible matrix over such a ring in the form $E \cdot D(\mu)$ because the g.c.d. of the augmentations of the elements of every column must be 1, so we can get an element with augmentation 1 in every column and then in every row after multiplying the matrix by appropriate elementary matrices. Thus the determinant in this case exists, and we can prove that it is unique and satisfies necessary properties by embedding $\mathbb{Z}G$ into $\mathbb{Q}G$ and using the standard argument.

Since the Jacobian matrix of any automorphism of F is invertible over $\mathbb{Z}F$ (see [6]), our Proposition 3.1 yields necessary condition(s) for an endomorphism of the group F to be an automorphism.

We now show that all the elements u_m in the statement of Proposition 3.1 can be taken equal to some fixed element u which depends only on the matrix M. This u can therefore be considered the "actual" determinant of M, and the elements u_m — its "approximations".

Proposition 3.2. *Under the conditions of Proposition* 3.1*, there exists an element* $u \in \mathbf{K}G$ *such that in the equality* (3)*, one has for any* $m \geqslant 2$*:* $u = u_m s_m + v_m$ *for some* $s_m \in P_m$*;* $v_m \in \Delta^m$*.*

Proof. Having in mind the product rule for the determinants, all we have to do is to show that the product of any two elements of the form of the right-hand side of (3) has again this form. Fix m, and let $u_1, u_2 \in \mathbf{K}G$; $p_1, p_2 \in P_m$; $w_1, w_2 \in \Delta^m$; then $(u_1p_1 + w_1)(u_2p_2 + w_2) = u_1p_1u_2p_2 + v$, where $v \in \Delta^m$. Now, $u_1p_1u_2p_2 = u_1u_2(u_2^{-1}p_1u_2)p_2$, so we only have to show that a conjugate of an element from P_m is again in P_m. This is clear for generating elements (2), and hence or any element of P_m as well. ☐

Corollary 3.3. *For a matrix* $M \in \mathrm{GL}_n(\mathbb{Z}F)$*, we have for any* $m \geqslant 2$*:*

$$\det_m(M) = \pm g \cdot p_m + w_m$$

for some $g \in F$*;* $p_m \in P_m$*;* $w_m \in \Delta^m$*.*

The following example is meant to illustrate the difference between the usual determinant of the abelianized matrix and our non-commutative determinant.

Example. Let the endomorphism ψ of the free group F_3 take $\{x_1, x_2, x_3\}$ to $[x_1[x_3, x_2, x_1, x_1], x_2, x_3]$. Then the Jacobian matrix J_ψ has zeroes below the diagonal and units on the diagonal except for the element $a_{11} = 1 + x_1 \cdot d_1([x_3, x_2, x_1, x_1])$.

The image of a_{11} in the ring $\mathbb{Z}(F/F')$ equals 1, hence the determinant of the matrix $\sigma_{F'}(J_\psi)$ also equals 1, in particular, this matrix is invertible (which implies that ψ induces an automorphism of the free metabelian group F/F'').

On the other hand, let us consider the image of J_ψ over the ring $\mathbb{Z}F/\Delta^4$ and compute $\delta_4(J_\psi)$:

$$\delta_4(J_\psi) = 1 - (1 - [x_3, x_2, x_1] + (x_1 - 1)(x_3 - 1)(x_2 - 1) - (x_1 - 1)(x_2 - 1)(x_3 - 1)) + \Delta^4.$$

If ψ were an automorphism of F, we would have by Corollary 3.3:

$$1 - [x_3, x_2, x_1] + (x_1 - 1)(x_3 - 1)(x_2 - 1) - (x_1 - 1)(x_2 - 1)(x_3 - 1) = \pm g \cdot p_4 \pmod{\Delta^4} \quad (5)$$

for some $g \in F$ and $p_4 \in P_4$. To get some information on what this element p_4 may look like, open the brackets in the expression (2) for the generators of P_m:

$$1 + vw - wv + v^2w - vwv + wvw - w^2v + \dots \tag{6}$$

Thus we see that monomials on $(x_i - 1)$ of weight 2 and 3 in the expansion (6) should belong to (Δ, Δ), the *additive subgroup* of $\mathbb{Z}F$ generated by elements of the form $uv - vu$; $u, v \in \Delta$. Hence the same is true for the expansion of any element from P_m.

Now (5) and (6) imply: $1 - (\pm g) = (x_1 - 1)(x_3 - 1)(x_2 - 1) - (x_1 - 1)(x_2 - 1)(x_3 - 1) \pm g \cdot w \pmod{\Delta^4}$ for some $w \in (\Delta, \Delta)$. It follows that $g - 1 \in \Delta^2$, so $g \in F'$, hence by Lemma 2.5, $g - 1 \in (\Delta, \Delta) + \Delta^3$. Also, $gw = w \pmod{\Delta^4}$. Using Lemma 2.5 again, we see that for any $h \in \gamma_3(F)$, one has $h - 1 \in (\Delta, \Delta) + \Delta^4$. Hence if we write the element g as a product of powers of basic commutators $[x_3, x_2]$, $[x_3, x_1]$ and $[x_2, x_1]$ modulo $\gamma_3(F)$, this will give $g - 1 = c_1([x_3, x_2] - 1) + c_2([x_3, x_1] - 1) + c_3([x_2, x_1] - 1) \pmod{(\Delta, \Delta) + \Delta^4}$ for some integers c_1, c_2, c_3.

It is clear that the expansion of any $([x_i, x_j]-1)$ modulo ? involves monomials in x_i and x_j only. All that yields:

$$(x_1 - 1)(x_3 - 1)(x_2 - 1) - (x_1 - 1)(x_2 - 1)(x_3 - 1) \in (\Delta, \Delta) + \Delta^4,$$

which is not the case. This contradiction proves that ψ is not an automorphism.

This proof of ψ being not an automorphism may seem somewhat delicate — this is because ψ is actually an automorphism "modulo F''", i.e., ψ indices an automorphism of the group F/F''. Thus we see that the usual commutative determinant (of the abelianized Jacobian matrix) is only good for distinguishing automorphisms modulo F'' whereas a non-commutative determinant allows a more subtle analysis.

Procedure. Given a square matrix M over an arbitrary group algebra $\mathbf{K}G$, we want to detect its non-invertibility. First we consider the matrix $\sigma(M)$ of the augmentations of elements of M. If the matrix $\sigma(M)$ is not invertible over $\mathbf{K}$, then we are done. If it is, it means we have at least one element with non-zero augmentation in every row and in every column of the matrix M; hence we can proceed with the computation of $\det_m(M)$ starting with $m = 1$ and going on until we have the condition (3) contradicted. The main point is, of course, the checking whether or not this condition is contradicted; it is hard possible to give a universal recipe here. For some "smooth" group like a free group, for example, the procedure sketched in the Example above, usually gives good results. In the case of integral group ring $\mathbb{Z}F$, Chen–Fox–Lyndon identities from [10] are useful in proving that given element of $\mathbb{Z}F$ is not equal to an element of F modulo Δ^m.

Now we are going to present a version of Proposition 3.1 which will play the crucial role in Section 4. Computing determinants modulo ? is very convenient, but in some special cases it is possible to get more subtle information upon computing determinants modulo powers of other, smaller, ideals.

First we make a very general observation which is valid for non-square matrices as well: we can consider decompositions of such a matrix in a product of elementary and diagonal matrices, where elementary matrices are the same as usual (in particular, they are square matrices), whereas diagonal matrices are those that have the same size as a matrix under consideration, and have 0 outside the (i,i)-th places.

Proposition 3.4. *Let S be an arbitrary ring with unit and without zero divisors. Let M be an arbitrary (not necessarily square) matrix over S. Suppose $M = E_1 \cdot D_1 = E_2 \cdot D_2$, where E_1 and E_2 are products of elementary matrices, and D_1, D_2 are diagonal matrices (in the sense defined above). Then there is a right invertible diagonal (square) matrix D such that $D_1 = D \cdot D_2$.*

Proof. Denote $E = E_2^{-1}E_1$; then $E \cdot D_1 = D_2$, and $E^{-1} \cdot D_2 = D_1$. Hence for every possible i, there are $u_i, v_i \in S$ such that $u_i \cdot d_1(ii) = d_2(ii)$ and $v_i \cdot d_2(ii) = d_1(ii)$, where $d_j(ii)$ denotes appropriate diagonal element of the matrix D_j. Since S has no zero divisors, this means the elements $v_i \cdot u_i$ are right invertible, hence v_i are right invertible, too. The result follows. □

For invertible square matrices, this can be "a little" improved:

Proposition 3.5. *Under the conditions of Proposition* 3.4*, let S have the property* (*) *(see the beginning of this section). Furthermore, let M be an* invertible square matrix*. Then a product (in an arbitrary order) of all diagonal elements of the matrix D from the statement of Proposition* 3.4 *belongs to the commutator subgroup $[S^*, S^*]$ of the multiplicative group S^*.*

Proof. Applying Proposition 3.4 shows that the matrix D is invertible (since both D_1 and D_2 should be invertible). Therefore, all the diagonal elements of D belong to the group S^*. Furthermore, D is a product of elementary matrices since $D = E_1^{-1}E_2$. Since S has the property (*), an argument from [1] (see also the beginning of this section) shows that the determinant of D (in the sense of Dieudonné) equals $\bar{1}$ in the group $S^*/[S^*, S^*]$. The result follows. □

4 Non-tame automorphisms

In this section, we are going to give some applications of non-commutative determinants to detecting non-tame automorphisms or generating systems of several groups.

Theorem 4.1. *Let $M_3 = F_3/F_3''$, the free metabelian group of rank* 3*, and $M_{3,2} = F_3/(F_3')^2$, the free* 2*-metabelian group of rank* 3*. The mapping $x_1 \to x_1[x_3, x_2, x_1, x_1]$; $x_2 \to x_2$; $x_3 \to x_3$, induces:*

(i) (*cf.* [9]) *a non-tame automorphism of M_3;*

(ii) *a non-tame automorphism of $M_{3,2}$.*

Proof. Of course, (i) follows from (ii), but it will be convenient for us to make the proof of (i) a part of the proof of (ii), so we begin with (i). Suppose by means of contradiction that for some $u_i \in F''$ we have an automorphism ϕ of the group F induced by $x_1 \to x_1[x_3, x_2, x_1, x_1]u_1$; $x_2 \to x_2u_2$; $x_3 \to x_3u_3$. Denote $R = F'$. Then diagonal elements of the Jacobian matrix J_ϕ have the form $1 + w$, $w \in \Delta_R$ (see computation in the Example in the previous section); in particular, they are all invertible modulo Δ_R^2. All the other elements of J_ϕ except those in the first row, belong to Δ_R. This allows us to reduce the matrix J_ϕ to a diagonal form by applying elementary transformations to its rows starting with the last row and going first up, then down with annihilations. In the end, we get an element of the following form in the upper left corner: $a_{11} = 1-\big(1-[x_3, x_2, x_1]+(x_1-1)(x_3-1)(x_2-1)-(x_1-1)(x_2-1)(x_3-1)-\sum d_i(u_i)\big)$ $(\mathrm{mod}\ \Delta^4)$ for some $u_i \in F''$, $i = 1, 2, 3$.

Consider now the ring $S = \mathbb{Z}F/\Delta_R^2$. Note that elements from S^* are of the form $\pm h + w + \Delta_R^2$ with $h \in F$, $w \in \Delta_R$. Hence elements from $[S^*, S^*]$ have the form $g + \Delta_R^2$ with $g \in F'$ (see (2)). Applying Proposition 3.5 then gives (note that $\Delta_R^2 \subseteq \Delta^4$):

$$1 - [x_3, x_2, x_1] + (x_1 - 1)(x_3 - 1)(x_2 - 1) - (x_1 - 1)(x_2 - 1)(x_3 - 1) - \sum d_i(u_i) = 1 - g \pmod{\Delta^4}. \quad (7)$$

Denote the element on the left-hand side of (7) by t. Obviously, $t \in \Delta^3$; hence $(g - 1) \in \Delta^3$, and $g \in \gamma_3(F)$ by Lemma 2.2 (iii). Hence by Lemma 2.5, $t \in (\Delta, \Delta) + \Delta^4$. Since $1 - [x_3, x_2, x_1] \in (\Delta, \Delta) + \Delta^4$ (again by Lemma 2.5), this leaves us with

$$(x_1-1)(x_3-1)(x_2-1)-(x_1-1)(x_2-1)(x_3-1)-\sum d_i(u_i) \in (\Delta, \Delta)+\Delta^4. \quad (8)$$

The element $s = (x_1 - 1)(x_3 - 1)(x_2 - 1) - (x_1 - 1)(x_2 - 1)(x_3 - 1)$ does not belong to $(\Delta, \Delta)+\Delta^4$, hence we have to compensate it by $\sum d_i(u_i)$. Since we are only interested in monomials of weight 3 in the expansion of $\sum d_i(u_i)$, we can consider u_i products of basic commutators of weight 4 from F''. Given the collection of generators in the monomials that we have to compensate, we are left with the following possibility: $u_1 = [[x_2, x_1], [x_3, x_1]]^{\varepsilon_1}$; $u_2 = [[x_3, x_2], [x_2, x_1]]^{\varepsilon_2}$; $u_3 = [[x_3, x_2], [x_3, x_1]]^{\varepsilon_3}$ for some integers $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Consider

$$\begin{aligned} d_1(u_1) = {} & \varepsilon_1((x_2 - 1)(x_1 - 1)(x_3 - 1) - (x_3 - 1)(x_1 - 1)(x_2 - 1) - \\ & (x_1 - 1)(x_2 - 1)(x_3 - 1) + (x_1 - 1)(x_3 - 1)(x_2 - 1)) = \\ & 2\varepsilon_1((x_3 - 1)(x_2 - 1)(x_1 - 1) - (x_2 - 1)(x_3 - 1)(x_1 - 1)) \pmod{(\Delta, \Delta) + \Delta^4}. \end{aligned}$$

Thus we see that all monomials in the expansion of $d_1(u_1)$ modulo $(\Delta, \Delta) + \Delta^4$ occur with even coefficients, hence they cannot compensate t. Expansions of $d_2(u_2)$ and $d_3(u_3)$ have the same form, and this completes the proof of (i).

Proof of (ii) goes along the same lines, but we consider the image of the Jacobian matrix J_ϕ over the ring $\mathbb{Z}_2F$. Furthermore, the elements u_i this time can belong to $(F')^2 = (F')^2 F''$, not just F''. As we have seen when proving part (i) of this theorem, Fox derivatives of elements from F'' don't contribute anything to compensate the element t modulo $(\Delta, \Delta) + \Delta^4$ (in the ring $\mathbb{Z}_2 F$). Hence let us consider u_i in the form $[x_3, x_2]^{2\varepsilon_1}[x_3, x_1]^{2\varepsilon_2}[x_2, x_1]^{2\varepsilon_3} \cdot c^2$ for some integers $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and $c \in \gamma_3(F)$. Then, $d_i(c^2) = (1+c)d_i(c) = 0 \pmod{\Delta^4}$ in $\mathbb{Z}_2 F$, and if an element has the form $[x_i, x_j]^m$, then the expansion modulo Δ^4 of any of its derivatives involves monomials on $(x_i - 1)$ and $(x_j - 1)$ only. Hence there is no way that $d_i(u_i)$ can compensate t. This completes the proof of the theorem. □

For the next application of our method, let us bear in mind that by $\gamma_c^{(p)}(F)$ we denote the c-th term of the Jennings series of F, p a prime (see Lemma 2.2). The following Theorem 4.2 can be considered a "modular" analog of the main result of [19].

Theorem 4.2. *Let R be a normal subgroup of the group F, and let $R \subseteq \gamma_c^{(p)}(F)$, but $R \nsubseteq \gamma_{c+1}^{(p)}(F)$ for some $c \geqslant 3$, $p \geqslant 2$ a prime. Suppose there is an element $h \in R$, $h \notin \gamma_{c+1}^{(p)}(F)$, which does not depend on some free generator, say, on x_1. Then the group F/R^pR' has non-tame generating systems.*

Proof. Consider the following generating system of F/R^pR': $[x_1[x_1, h, x_1], x_2, \ldots, x_n]$ (we denote elements of the group F and their natural images in F/R^pR' by the same letters without ambiguity). Were this generating system tame, for same $u_i \in R^pR'$ we would have an automorphism ϕ of F given by $\phi\colon x_1 \to x_1[x_1, h, x_1]u_1,\ x_i \to x_i u_i,\ 2 \leqslant i \leqslant n$. Denote $y_i = \phi(x_i)$ and compute the elements of the Jacobian matrix J_ϕ over the group ring $\mathbb{Z}_p F$. Since $R \subseteq \gamma_3^{(p)}(F)$, we have $\Delta_{R^pR'} \subseteq \Delta^{c+3}$, hence:

$$
\begin{aligned}
&d_1(y_1) = 1 + (h-1)(x_1 - 1) + ([x_1, h] - 1) \pmod{\Delta^{c+2} \cap \Delta_R};\\
&d_i(y_1) = -(x_1 - 1)^2 d_i(h) \pmod{\Delta^{c+2} \cap \Delta_R} \ \text{ for } \ i \geqslant 2;\\
&d_i(y_i) = 1 \pmod{\Delta^{c+2} \cap \Delta_R} \ \text{ for } \ i \neq 1;\\
&d_i(y_j) = 0 \pmod{\Delta^{c+2} \cap \Delta_R} \ \text{ for } \ i \neq 1,\ i \neq j.
\end{aligned}
$$

Thus we can diagonalize the matrix J_ϕ over the ring $S = \mathbb{Z}_p F/\Delta_R^2$ by applying elementary transformations to its rows, and then apply Proposition 3.5. This gives:

$$1 + (h-1)(x_1 - 1) = g \pmod{\Delta^{c+2} + \Delta_R^2}. \tag{9}$$

Since $R \subseteq \gamma_c^{(p)}(F)$, (9) implies $g - 1 \in \Delta^{c+1}$, so $g \in \gamma_{c+1}^{(p)}(F)$. Hence we are in a position to apply Lemma 2.4 (ii) with $i_1 = 1$:

$$\sigma(d_{1i_2\ldots i_{c+1}}(g)) + \sigma(d_{i_2 1\ldots i_{c+1}}(g)) + \sigma(d_{i_2 i_3 \ldots i_{c+1} 1}(g)) = 0, \tag{10}$$

where $i_2, \ldots, i_{c+1}$ may run through $\{2, \ldots, (c+1)\}$.

From (9), we see that $d_{1i_2\ldots i_{c+1}}(g) = d_{i_2\ldots i_{c+1}}(h)$, hence for some set of indices $i_2, \ldots, i_{c+2}$, we should have $\sigma(d_{1i_2\ldots i_{c+1}}(g)) \neq 0$ in $\mathbb{Z}_p F$. On the other hand, $\sigma(d_{1i_2\ldots i_{c+1}}(g))$ is the only possibly non-zero summand in (10). This contradiction completes the proof of Theorem 4.2. □

In conclusion, we consider liftings of automorphisms of some F/R groups to automorphisms of F/R' groups.

Theorem 4.3. *For any $c \geqslant 3$ and $n \geqslant 2$, the following automorphism ϕ of the group $F/\gamma_{c+1}(F)$ cannot be lifted to an automorphism of $F/(\gamma_{c+1}(F))'$: $\phi\colon x_1 \to x_1[x_1, x_2, x_2],\ x_i \to x_i,\ 2 \leqslant i \leqslant n$.*

Proof. Let $R = \gamma_{c+1}(F)$. By the result of [14], the matrix $M = \sigma_R(J_\phi)$ belongs to $\mathrm{GL}_n(\mathbb{Z}(F/R))$. Then, by a result of [7], we can extend the matrix M to a matrix M' by placing 1 on the diagonal and 0 elsewhere so that M' would be a product of elementary and diagonal matrices, hence Proposition 3.5 applies to the matrix M'. Let M' be an $m \times m$ matrix, $m \geqslant n$.

If the automorphism ϕ were tame, the extension ϕ' of ϕ to the free nilpotent group of rank m by setting $\phi'(x_i) = x_i$, $n \leqslant i \leqslant m$, would be also a tame automorphism, and $M' = \sigma_R(J_{\phi'})$.

Now set $S = \mathbb{Z}F/\Delta_R = \mathbb{Z}(F/R)$; it is well-known that invertible elements of the ring S are of the form $\pm h + \Delta_R$ with $h \in F$; hence elements from $[S^*, S^*]$ have the form $g + \Delta_R$ with $g \in F'$. Applying now our Proposition 3.5 to M' yields: $1 + (x_2 - 1)^2 = g \pmod{\Delta_R}$. Since $\Delta_R \subseteq \Delta^3$, this implies

$$(x_2 - 1)^2 = g - 1 \pmod{\Delta^3}, \tag{11}$$

$g \in F'$. But this is impossible since by Lemma 2.4 (i), we should have $2 \cdot \sigma(d_{22}(g)) = (\sigma(d_2(g)))^2 - \sigma(d_2(g)) = 0$ because $g \in F'$. However (11) gives $\sigma(d_{22}(g)) = 1$, a contradiction. □

In the same manner we can prove that the automorphism of the free metabelian group of rank 3 considered in Theorem 4.1, cannot be lifted to an automorphism of F_3/F_3''':

Theorem 4.4. *The automorphism ϕ of the group $M_3 = F_3/F_3''$ given by $x_1 \to x_1[x_3, x_2, x_1, x_1]$; $x_2 \to x_2$; $x_3 \to x_3$, cannot be lifted to an automorphism of the group $S_{3,3} = F_3/F_3'''$.*

Proof. Proceeding as in the proof of the previous theorem, we get the image of the Jacobian matrix J_ϕ over the group ring $\mathbb{Z}(M_3)$; denote this image by B. Then, take the image M of the matrix B over $\mathbb{Z}(F/\gamma_4(F))$ (bear in mind that $F'' \subseteq \gamma_4(F)$); this matrix M must belong to $\mathrm{GL}_3(\mathbb{Z}(F/\gamma_4(F)))$. Extend this matrix to a matrix M' which is a product of elementary and diagonal matrices (by [7]); then our Proposition 3.5 is applicable to M'; this yields the same inclusion (8) as in the proof of Theorem 4.1:

$$(x_1 - 1)(x_3 - 1)(x_2 - 1) - (x_1 - 1)(x_2 - 1)(x_3 - 1) - \sum d_i(u_i) \in (\Delta, \Delta) + \Delta^4.$$

Applying the argument from the proof of Theorem 4.1 yields the result. □

As regards free metabelian groups of rank $n \neq 3$, they have only tame automorphisms (see [2], [3]), so in particular, they have every automorphism induced by an automorphism of F_n/F_n'''. Summing up these results with a result of [19], we have the following complete answer to Problem 5.49 (b) from [13]:

Corollary 4.5. *Let $S_{n,m}$ denote the free solvable group of rank $n \geqslant 2$ and derived length $m \geqslant 1$. Every automorphism of $S_{n,m}$ is induced by an automorphism of $S_{n,m+1}$ in the following and only in the following cases:* (i) $m = 1$, $n \geqslant 2$; (ii) $m = 2$, $n \neq 3$.

References

[1] E. Artin, *Geometric Algebra*, Interscience, New York (1957).

[2] S. Bachmuth, E. Formanek, and H. Y. Mochizuki, IA-automorphisms of certain two-generator torsion-free groups, *J. Algebra*, **40** (1976), 19–30.

[3] S. Bachmuth and H. Y. Mochizuki, $\mathrm{Aut}\,(F) \to \mathrm{Aut}\,(F/F'')$ is surjective for free group F of rank $\geqslant 4$, *Trans. Amer. Math. Soc.*, **292** (1985), 81–101.

[4] S. Bachmuth and H. Y. Mochizuki, The tame range of automorphism groups and GL_n, in *Group Theory (Singapore, 1987)*, De Gruyter, Berlin–New York (1989) 241–251.

[5] H. Bass, *Algebraic K-theory*, W. A. Benjamin, New York (1968).

[6] J. S. Birman, An inverse function theorem for free groups, *Proc. Amer. Math. Soc.*, **41** (1973), 634–638.

[7] K. A. Brown, T. H. Lenagen, and J. T. Stafford, K-theory and stable structure of some noetherian group rings, *Proc. London Math. Soc.*, **42** (1981), 193–230.

[8] R. M. Bryant, C. K. Gupta, F. Levin, and H. Y. Mochizuki, Non-tame automorphisms of free nilpotent groups, *Comm. Algebra*, **18** (11) (1990), 3619–3631.

[9] O. Chein, IA automorphisms of free and free metabelian groups, *Comm. Pure Appl. Math.*, **21** (1968), 605–629.

[10] K. T. Chen, R. H. Fox, and R. C. Lyndon, Free differential calculus. IV. The quotient groups of the lower central series, *Ann. Math.*, **68** (1958), 81–95.

[11] P. M. Cohn, *Free Rings and Their Relations*, Academic Press, London, second edition edition (1985).

[12] R. H. Fox, Free differential calculus. I. Derivation in the free group ring, *Ann. Math. (2)*, **57** (1953), 547–560.

[13] *The Kourovka Notebook*, volume 121 of *Translations of the AMS* (1983).

[14] A. F. Krasnikov, Generators of the group $F/[N,N]$, *Math. Notes*, **24** (1979), 591–594.

[15] M. Lustig and Y. Moriah, Generating systems of groups and Reidemeister–Whitehead torsion, *J. Algebra*, **157** (1993), 170–198.

[16] R. Lyndon and P. Schupp, *Combinatorial Group Theory*, volume 89 of *Series of Modern Studies in Math.*, Springer-Verlag, Berlin–Heidelberg–New York (1977).

[17] J. Nielsen, Die Isomorphismengruppe der freien Gruppen, *Math. Ann.*, **91** (1924), 169–209.

[18] D. S. Passman, *The algebraic structure of group rings*, John Wiley and Sons, New York (1977).

[19] V. Shpilrain, Automorphisms of F/R' groups, *Internat. J. Algebra Comp.*, **1** (1991), 177–184.