On the Andrews-Curtis equivalence

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ABSTRACT. The Andrews-Curtis conjecture claims that every balanced presentation of the trivial group can be reduced to the standard one by a sequence of "elementary transformations" which are Nielsen transformations augmented by arbitrary conjugations. It is a prevalent opinion that this conjecture is false; however, not many potential counterexamples are known. In this paper, we show that some of the previously proposed examples are actually not counterexamples. We hope that the tricks we used in constructing relevant chains of elementary transformations will be useful to those who attempt to establish the Andrews-Curtis equivalence in other situations.

On the other hand, we give two rather general and simple methods for constructing balanced presentations of the trivial group; some of these presentations can be considered potential counterexamples to the Andrews-Curtis conjecture. One of the methods is based on a simple combinatorial idea of composition of group presentations, whereas the other one uses "exotic" knot diagrams of the unknot.

We also consider the Andrews-Curtis equivalence in metabelian groups and reveal some interesting connections of relevant problems to well-known problems in K-theory.

1. Introduction

Let $F = F_n$ be the free group of a finite rank $n \geq 2$ with a set $X = \{x_1, ..., x_n\}$ of free generators. Let $Y = \{y_1, ..., y_m\}$ be an arbitrary finite tuple of elements of the group F. Consider the following elementary transformations that can be applied to Y:

(AC1) y_i is replaced by $y_i y_j$ for some $j \neq i$;

(AC2) y_i is replaced by y_i^{-1};

(AC3) y_i is replaced by gy_ig^{-1} for some $g \in F$.

It is understood that y_j does not change if $j \neq i$.

We say that two tuples Y and $\widetilde{Y}$ are Andrews-Curtis equivalent (AC-equivalent) if one of them can be obtained from another by applying a sequence of transformations (AC1)–(AC3).

Denote by $\langle Y \rangle^F$ the normal closure of a set Y in F. The following conjecture appears to be of interest in topology as well as group theory (see [2]):

The Andrews-Curtis conjecture. Let $Y = \{y_1, ..., y_n\}$ be a tuple of elements of $F = F_n$. Then $\langle Y \rangle^F = F$ if and only if Y is AC-equivalent to $X = \{x_1, ..., x_n\}$.

This also has an obvious interpretation in a different language: every balanced presentation of the trivial group can be reduced to the standard one by a sequence of transformations (AC1)–(AC3). We shall say that two presentations of a group are AC-equivalent if the sets of relators in these presentations are AC-equivalent.

A weaker form of this conjecture (which is of greater interest in topology) arises when one allows two more elementary transformations:

(AC4) Y is extended to $\{y_1, ..., y_m, x_\nu\}$, where x_ν is a new free generator (i.e., $y_1, ..., y_m$ do not depend on x_ν).

(AC5) the converse of (AC4).

If $\widetilde{Y}$ can be obtained from Y by means of transformations (AC1–AC5), we say that $\widetilde{Y}$ is *stably* AC-equivalent to Y.

The weak Andrews-Curtis conjecture. For a tuple $Y = \{y_1, ..., y_n\}$, one has $\langle Y \rangle^F = F$ if and only if Y is stably AC-equivalent to $X = \{x_1, ..., x_n\}$.

We mention here a connection of these conjectures to 2-dimensional homotopy theory. There is a standard way of constructing a 2-dimensional cell complex from an arbitrary finite presentation of a group (see e.g. [16, p.117]). (A complex like that has a single vertex; the number of 1-dimensional cells (edges) equals the number of generators in a given group presentation, and the number of 2-dimensional cells equals the number of defining relators). Two cell complexes are called simple-homotopy equivalent if one of them can be obtained from another by a sequence of elementary expansions and collapses (where "elementary" means that only one cell at a time is added/cancelled).

It is well-known (see e.g. [27] for details) that (stably) AC-equivalent sets of relators give rise to simple-homotopy equivalent complexes, since the transformations (AC1-AC5) applied to defining relators of the fundamental group of a 2-complex, are mimicked by elementary expansions and collapses applied to this 2-complex. The point however is that those elementary expansions/collapses that mimic the AC-transformations, only involve cells of

dimension 3 or lower, whereas to establish a simple-homotopy equivalence of two 2-complexes, one is allowed to use cells of arbitrary dimensions.

Now, it is known that any two 2-complexes with the trivial fundamental group (i.e., 2-complexes contractible to a point) are simple-homotopy equivalent, and by a result of Whitehead [26], one does not need to use cells of dimension higher than 4 to establish a simple-homotopy equivalence in that situation.

So, the point of the (topological) Andrews-Curtis conjecture is that one does not actually have to use cells of dimension higher than 3 when treating contractible 2-complexes. This is equivalent to the weak Andrews-Curtis conjecture above (see [27]).

We also note that the Andrews-Curtis conjecture admits several generalizations – see the surveys [10], [12], [13], [19]. Here we would like to point out the following very natural generalization.

Let G be a group and k a positive integer. We say that G *satisfies the Andrews-Curtis property* ACP_k if for any two k-tuples $U, V \in G^k$ the following equivalence holds:

$$\langle U \rangle^G = \langle V \rangle^G = G \Longleftrightarrow U \sim V,$$

where $U \sim V$ means AC-equivalence. (Note that the definition of AC-equivalence can be carried on to arbitrary groups.)

Question. *Is it true that for every group G, there is an integer k_0 such that G satisfies ACP_k for every $k \geq k_0$?*

Note that if A is a finite abelian group represented as

$$A \simeq \mathbf{Z}_{e_1} \times \cdots \times \mathbf{Z}_{e_r},$$

where $e_1 \mid e_2 \mid \ldots \mid e_r$, then A does not satisfy ACP_r, but it does satisfy ACP_k for every $k > r$. This result is due to B. H. Neumann and H. Neumann [24] (see also [8]).

For some groups, the above question is known to have the affirmative answer (see [22] or discussion below), but we are unaware of any example of a group where the answer would be negative.

It is a prevalent opinion that the Andrews-Curtis conjecture is false; however, not many potential counterexamples are known, the reason being the difficulty of constructing non-trivial (balanced) presentations of the trivial group. Here we give a (probably, partial) list of previously known potential counterexamples:

(1) $\langle x, y \mid x^{-1} y^2 x = y^3, \ y^{-1} x^2 y = x^3 \rangle$.
(2) $\langle x, y, z \mid y^{-1} xy = x^2, \ z^{-1} yz = y^2, \ x^{-1} zx = z^2 \rangle$.
(3) $\langle x, y \mid x^4 y^3 = y^2 x^2, x^6 y^4 = y^3 x^3 \rangle$.
(4) $\langle x, y \mid x^n = y^{n+1}, \ xyx = yxy \rangle, \ n \geq 2$.

The first two examples are probably the most established ones; see [7] for discussion. The presentation (3) is attributed to M.Wicks (by folklore), but we do not have a relevant reference. The series (4), which comes from [1], is valuable since it provides an infinite collection of examples. The following series has appeared recently [20]:

(5) $\langle x, y|\ x^{-1}y^n x = y^{n+1},\ x = w \rangle$, where $n \geq 1$, and w is a word in x and y with exponent sum 0 on x.

Note that the series (5) contains the example (1), and, furthermore, Proposition 1.3 below shows that every presentation in the series (4) is AC-equivalent to a presentation in the series (5).

Finally, the following series, due to C. Gordon (see [5]), was brought to our attention by W. Metzler.

(6) $\langle x, y|\ x = [x^m, y^n],\ y = [x^p, y^q] \rangle$, where m, n, p, q are arbitrary integers.

It is natural to start out by trying potential counterexamples with minimal total length of relators. Assisted by a computer, we were able to "crack" all potential counterexamples with total length of up to 12. More precisely (see [23]): any presentation of the trivial group of the form $\langle x, y\ |\ r(x,y), s(x,y) \rangle$ with $|r(x,y)| + |s(x,y)| \leq 12$ is AC-equivalent to the standard one.

We also mention here an auxiliary result of independent interest (see [23]): if G is a group defined by a presentation $\langle\ x, y\ |\ r(x,y),\ s(x,y)\ \rangle$, where $|r(x,y)| + |s(x,y)| \leq 12$, and the abelianization of G is trivial, then G is either the trivial group or isomorphic to the following finite group of order 120: $\langle x, y\ |\ yxy = x^2,\ xyx = x^4\ \rangle$.

These results were obtained by using MAGNUS software package for symbolic computation.

In this paper, we treat a couple of most interesting (in our opinion) presentations covered by the result of [23] cited above, because we hope that the tricks we used in constructing relevant chains of elementary transformations will be useful to those who attempt to establish the Andrews-Curtis equivalence in other situations:

Proposition 1.1. [23] Each of the following presentations of the trivial group is AC-equivalent to $\langle x, y|\ x, y \rangle$:

(a) $\langle x, y|\ x^2 = y^3,\ xyx = yxy \rangle$.
(b) $\langle x, y|\ x^{-1}yx = y^2,\ x = yx^2yx^{-2} \rangle$.
(c) $\langle x, y|\ x^{-1}y^2 x = y^3,\ x^2 = yxy^{-1} \rangle$.
(d) $\langle x, y|\ x^{-1}y^2 x = y^3,\ x^2 = yxy \rangle$.

We note that some time ago, S. Gersten (unpublished) showed that the presentation (a) is *stably* AC-equivalent to $\langle x, y|\ x, y \rangle$.

Thus, the minimal total length of relators in potential counterexamples

that still stand, is 13, as in

$$\langle x, y |\ x^3 = y^4,\ xyx = yxy \rangle.$$

Recently, Havas and Ramsay [11] showed that this is, in fact, the only (up to AC-equivalence) possible counterexample of length 13. We were not able to crack this example, but the following fact might be of interest:

Proposition 1.2. The presentation $\langle x, y |\ x^3 = y^4,\ xyx = yxy \rangle$ is stably AC-equivalent to $\langle x, y |\ x^4 = yx^2y^{-1}x^{-1}yx^2y^{-1},\ y = [x^2, y]^3 \rangle$. (Our commutator notation is $[a, b] = aba^{-1}b^{-1}$.)

This gives what seems to be the first example of two stably AC-equivalent sets in a free group, that are not known to be AC-equivalent. The longer presentation eludes all attempts to decrease the total length of relators by using the transformations (AC1)–(AC3) only. We have obtained this presentation by employing a trick that was originally used (in a different context) by McCool and Pietrowski [17]. Further examples of stably AC-equivalent sets that are not known to be AC-equivalent are provided by our Theorem 1.4 below, but they are based on an altogether different idea.

Also, we were able to show that every presentation in the series (4) above is AC-equivalent to a presentation in the series (5):

Proposition 1.3. The presentation $\langle x, y |\ x^n = y^{n+1},\ xyx = yxy \rangle$, $n \geq 2$, is AC-equivalent to $\langle x, y |\ x^{-1}y^n x = y^{n+1},\ x = y^{-1}x^{-1}yxy \rangle$.

All these examples are discussed in Section 2. In Section 3, motivated by the series (5) of balanced presentations of the trivial group, we give a similar generic series in the free group of rank 3. It is based on a rather general and simple method, namely, on obtaining balanced presentations of the trivial group from "exotic" knot diagrams of the unknot. This method can therefore produce numerous potential counterexamples to the Andrews-Curtis conjecture in a free group of any rank ≥ 2. (Note however that these are *not* counterexamples to the *weak* Andrews-Curtis conjecture since it is known that Reidemeister moves applied to a knot diagram give stably AC-equivalent Wirtinger presentations of the knot group.) In particular, we have:

Theorem 1.4. Any presentation of the following form determines the trivial group: $\langle x, y, z \mid x = z \cdot [y^{-1}, x^{-1}, z],\ y = x \cdot [y^{-1}, x^{-1}, z^{-1}] \cdot [z^{-1}, x],\ w \rangle$, where w is a word in x, y and z whose exponent sum on x, y and z equals ± 1.

We note that the normal closure of the first two relators contains the commutator subgroup of the ambient free group F_3 since the corresponding two-relator group is the fundamental group of the unknot, i.e., is infinite cyclic. Hence, by a sequence of AC-transformations, one can reduce w to a primitive element of F_3, and therefore reduce the 3-generator presentation to a 2-generator one. For example, if in the presentation in Theorem 1.4, we

take $w = x^{-1}yz$, then, after simplifications, we get the following 2-generator presentation with the total length of relators equal to 25:

$$\langle x, y \mid x^{-1}y^{-1}xy^{-1}x^{-1}yxy^{-2}xyx^{-1}y, \ y^{-1}x^{-1}y^2x^{-1}y^{-1}xyxy^{-2}x \rangle.$$

Our computer program was able to reduce the total length of relators to 14:

$$\langle x, y \mid xyx^{-2}y^{-1}xy^{-1}, \ x^{-1}y^{-1}xy^2xy^{-1} \rangle,$$

but this latter presentation eludes all attempts on decreasing the length.

We also note that the method of constructing presentations of the trivial group based on "exotic" knot diagrams of the unknot, is very flexible: by changing various parameters of the method, one can obtain essentially different presentations from the same knot diagram – see the remark after the proof of Theorem 1.4. Moreover, there is a lot of freedom in choosing the word w, and therefore in constructing potential counterexamples of rank $n \geq 2$ based on generic presentations of rank $n + 1$.

In Section 4, we give yet another procedure for constructing balanced presentations of the trivial group by using a composition method to generate new balanced presentations from old ones. We realize that this procedure may be known, but we were unable to find it in the literature.

Briefly, the construction is as follows. Let

$$P = \langle x_1, \ldots, x_n \mid r_1(X), \ldots, r_n(X) \rangle = \langle X \mid R \rangle,$$

$$Q = \langle x_1, \ldots, x_n \mid s_1(X), \ldots, s_n(X) \rangle = \langle X \mid S \rangle$$

(where $r_i(X)$, $s_i(X)$ are words in generators $x_1, \ldots, x_n$) be two balanced presentations of the trivial group. We assume here that relators form an ordered set, so that reordering the relators results in a different presentation. Define a new presentation $P \circ Q$ as follows:

$$P \circ Q = \langle x_1, \ldots, x_n \mid r_1(S), \ldots, r_n(S) \rangle = \langle X \mid R(S) \rangle,$$

where $r_i(S)$ is the word obtained from $r(X)$ upon replacing x_j by s_j for all j.

Then, if $P = \langle X \mid R \rangle$ and $Q = \langle X \mid S \rangle$ were balanced presentations of the trivial group, the new presentation $P \circ Q$ is that of the trivial group as well (see Lemma 4.1).

It is easy to see that the set of all balanced presentations on generators $x_1, \ldots, x_n$ forms a semigroup $\mathcal{P}_n$ with respect to the composition $\circ$. Clearly, $\mathcal{P}_n$ is embeddable into the semigroup of all endomorphisms of F_n. Note that every finite subset $K \subset \mathcal{P}_n$ generates a series of new presentations – those in the semigroup $\langle K \rangle$ generated by K in $\mathcal{P}_n$. This allows one to produce easily new examples of balanced presentations of the trivial group. The following result shows that the Andrews-Curtis conjecture holds for every presentation from $\mathcal{P}_n$ if and only if it holds for every presentation from some generating set of $\mathcal{P}_n$.

Proposition 1.5. (cf. [21]) Let K be a subset of $\mathcal{P}_n$. Then every presentation in the semigroup $\langle K \rangle$ generated by K in $\mathcal{P}_n$ satisfies the Andrews-Curtis conjecture if and only if every presentation in K does.

In Section 5, we consider the Andrews-Curtis equivalence in metabelian groups of the form $F/[R, R]$, where R is a normal subgroup of a free group F. (The definition of the AC-equivalence generalizes obviously to arbitrary groups.) We shall denote elements of a free group and their natural images in a group $F/[R, R]$ by the same letters when there is no ambiguity.

Myasnikov [22] settled the analog of the Andrews-Curtis conjecture for those groups in the positive. More precisely, he showed (in the case where F/R is a free abelian group of rank $n \geq 2$) that if, in a metabelian group $M = F/[R, R]$, there are two sets of cardinality n whose normal closures equal M, then those sets are AC-equivalent. (In fact, his result is somewhat more general and remains valid, in particular, for free solvable groups of arbitrary derived length.)

In sharp contrast, we prove here the following

Theorem 1.6. Let $F = F_n$ be the free group of rank $n \geq 3$ generated by $X = \{x_1, ..., x_n\}$, and let R be the normal closure of $Y = \{[x_1, x_2], x_3\}$. Then, in the metabelian group $M = F/[R, R]$, there are infinitely many pairwise AC-inequivalent, but stably AC-equivalent 2-element sets each of which has the same normal closure R in the group M. A particular pair of elements with the same normal closure as Y but AC-inequivalent to Y, is

$$\{[x_1, x_2][x_1, x_2, x_1]^{-2x_2^{-1}} \cdot x_3^{4x_2^{-1}}, \ [x_1, x_2, x_1, x_1]^{-x_2^{-1}} \cdot x_3[x_3, x_1]^{2x_2^{-1}}\}.$$

We note that Lustig [15] and Metzler [18] constructed (finitely many) pairwise AC-inequivalent sets of the same cardinality (≥ 3) in a free group, each of which has the same normal closure. It is not known however if an *infinite* family of sets like that exists in a free group. Furthermore, the sets constructed in [15] and [18] are also stably AC-inequivalent since the corresponding 2-complexes are simple-homotopy inequivalent. It seems to be another very difficult question as to whether or not there are stably AC-equivalent, but AC-inequivalent sets of the same cardinality in a free group.

It is therefore amazing that metabelian groups not only admit an easier solution of the problem analogous to the Andrews-Curtis conjecture, but also admit easier counterexamples to its generalization.

Our proof of Theorem 1.6 is based on the fact that there are non-tame invertible 2×2 matrices over Laurent polynomial rings in 2 or more variables. (An invertible square matrix is called *tame* if it is a product of elementary and diagonal matrices). This fact is due to Bachmuth and Mochizuki [3]. A particular matrix like that was recently found by Evans [9].

On the other hand, every invertible square matrix over a Laurent polynomial ring is *stably tame*, which means one can extend this matrix by placing 1 on the diagonal and 0 elsewhere, such that the extended matrix is tame. This follows from Suslin's stability theorem [25].

In view of the aforementioned results of Suslin and Bachmuth-Mochizuki, the following open problem is of particular interest (see [4] for a survey): *is it true that every matrix from $GL_2(\mathbf{Z}[t^{\pm 1}])$ is tame?* Here $\mathbf{Z}[t^{\pm 1}]$ denotes the integral group ring of the infinite cyclic group, i.e., the ring of Laurent polynomials in one variable. We show that this notorious problem in K-theory is also related to the AC-equivalence:

Proposition 1.7. Every matrix from $GL_2(\mathbf{Z}[t^{\pm 1}])$ is tame if and only if every set $\widetilde{Y}$ such that $< \widetilde{Y} >^M = < Y >^M = R$, is AC-equivalent to $Y = \{x_1, x_2\}$ in the group $M = F_3/[R, R]$.

2. AC-equivalent presentations

We start by listing two extra transformations that facilitate computations toward establishing the AC-equivalence of various sets. Either of them is a composition of several transformations (AC1)–(AC3). The first transformation comes from [7], where it is called "Basic substitution principle".

Making a substitution. If in a set $Y = \{y_1, ..., y_m\}$, one replaces some element y_i by an element $\tilde{y}_i$ which is congruent to y_i modulo the normal closure of $Y = \{y_1, ..., y_{i-1}, y_{i+1}, ..., y_m\}$, then the resulting set $\widetilde{Y}$ will be AC-equivalent to Y.

The second transformation is a composition of several (AC1)–(AC3) only if the normal closure of a given set Y is the whole group F.

Applying an automorphism. If $\langle Y \rangle^F = F$, then applying any automorphism of the ambient free group F to *every element* of the set Y gives a set which is AC-equivalent to Y.

This last statement has the following useful corollary:

Corollary 2.1. If $\langle x, y| \ r_1, r_2 \rangle$ is a presentation of the trivial group and r_1 is a primitive element of the ambient free group F_2, then this presentation is AC-equivalent to the standard one $\langle x, y| \ x, y \rangle$.

Now we are ready for a proof of Proposition 1.1. We shall just give a relevant chain of AC-transformations in each case (skipping obvious steps sometimes), emphasizing the most subtle steps in a chain by placing a (!) after them. (This is influenced by chess notation.)

Proof of Proposition 1.1.

(a) $\langle x, y| \ x^2 = y^3, xyx = yxy \rangle \longrightarrow \langle x, y| \ x^{-1}y^3 x^{-1}, yxy^{-1}x^{-1}y^{-1}x \rangle \longrightarrow$
$\langle x, y| \ x^{-1}y^3 x^{-1}, yxy^{-1}x^{-1}y^2 x^{-1} \rangle \longrightarrow \langle x, y| \ yx^{-1}y^3 x^{-1}y^{-1}, yxy^{-1}x^{-1}y^2 x^{-1} \rangle$
$\longrightarrow \langle x, y| \ yx^{-1}y^2 x^{-1}y^2 x^{-1}, \ yxy^{-1}x^{-1}y^2 x^{-1} \rangle.$

Now the first relator is a primitive element (say, the automorphism $x \to xy^2$; $y \to y$ takes it to $y^{-1}x^{-3}$), therefore we are done.

(b) $\langle x, y| \ x^{-1}yx = y^2, x = yx^2 yx^{-2} \rangle \longrightarrow \langle x, y| \ x^{-1}yx = y^2, x^{-1}yx^2 yx^{-2} \rangle \longrightarrow$
(substitution of y^2 for $x^{-1}yx$) $\langle x, y| \ x^{-1}yx = y^2, y^2 xyx^{-2} \rangle \longrightarrow$

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$\langle x,y|\ x^{-1}yx = y^2, x^{-1}y^2xyx^{-1}\rangle \longrightarrow$ (substitution of y^4 for $x^{-1}y^2x$)
$\langle x,y|\ x^{-1}yx = y^2, y^5x^{-1}\rangle$ (!).

Now the second relator is a primitive element, therefore we are done.

(c) $\langle x,y|\ x^{-1}y^2x = y^3, x^2 = yxy^{-1}\rangle \longrightarrow \langle x,y|\ y^{-2}x^{-1}y^2xy^{-1}, x^{-2}yxy^{-1}\rangle \longrightarrow$
$\langle x,y|\ y^{-2}x^{-1}y^2xy^{-1}, yx^{-1}y^{-1}x^2\rangle \longrightarrow \langle x,y|\ y^{-2}x^{-1}yx^2, yx^{-1}y^{-1}x^2\rangle \longrightarrow$
$\langle x,y|\ xy^{-2}x^{-1}yx, xyx^{-1}y^{-1}x\rangle \longrightarrow \langle x,y|\ xyxy^{-2}x^{-1}yxy^{-1}x^{-1}, xyx^{-1}y^{-1}x\rangle$
$\longrightarrow \langle x,y|\ xyxy^{-2}, xyx^{-1}y^{-1}x\rangle \longrightarrow \langle x,y|\ y^{-1}xyxy^{-1}, yx^{-1}y^{-1}x^2\rangle \longrightarrow$
$\langle x,y|\ y^{-1}x^3, yx^{-1}y^{-1}x^2\rangle$.

(d) $\langle x,y|\ x^{-1}y^2x = y^3,\ x^2 = yxy\rangle \longrightarrow \langle x,y|\ x^{-1}y^2x = y^3, xy^{-1} = x^{-1}yx\rangle \longrightarrow$
(substitution of $(xy^{-1})^2$ for $x^{-1}y^2x$) $\langle x,y|\ xy^{-1}xy^{-1} = y^3, xy^{-1} = x^{-1}yx\rangle$(!)
$\longrightarrow$ (applying the automorphism $x \to xy;\ y \to y$) $\langle x,y|\ x^2 = y^3, x = y^{-1}x^{-1}yxy\rangle \longrightarrow \langle x,y|\ x^2 = y^3, yxx = yxy\rangle$.

Now the result follows from part (a). $\square$

Proof of Proposition 1.2. By Proposition 1.3, $\langle x,y|\ x^3 = y^4,\ xyx = yxy\rangle$ is AC-equivalent to $\langle x,y|\ x^{-1}y^3x = y^4,\ xyx = yxy\rangle$, so we can start from there:
$\langle x,y|\ x^{-1}y^3x = y^4,\ xyx = yxy\rangle \longrightarrow (x \to x^{-1};\ y \to y^{-1})$
$\langle x,y|\ x^{-1}y^{-1}x^{-1} = y^{-1}x^{-1}y^{-1},\ xy^{-3}x^{-1} = y^{-4}\rangle \longrightarrow$
$\langle x,y|\ xyx = yxy,\ xy^{-3}x^{-1} = y^{-4}\rangle \longrightarrow (x \to xy^{-1};\ y \to y)$
$\langle x,y|\ x^2y^{-1} = yx,\ xy^{-3}x^{-1} = y^{-4}\rangle \longrightarrow \langle x,y|\ y^{-1}x^2 = xy,\ xy^{-3} = y^{-4}x\rangle \longrightarrow \langle x,y|\ y^{-1}x^2 = xy,\ xy^{-3} = y^{-4}xyy^{-1}\rangle \longrightarrow$ (substitution)
$\langle x,y|\ y^{-1}x^2 = xy,\ xy^{-3} = y^{-5}x^2y^{-1}\rangle$(!) $\longrightarrow \langle x,y|\ y^{-1}x^2 = xy,\ y^{-5}x^2 = xyy^{-3}\rangle \longrightarrow$ (substitution) $\langle x,y|\ y^{-1}x^2 = xy,\ y^{-4}x^2 = x^2y^{-3}\rangle \longrightarrow$
$\langle x,y|\ x^2 = yxy,\ y^{-4}x^2 = x^2y^{-3}\rangle \longrightarrow$
$\langle x,y,z|\ x^2 = yxy,\ y^{-4}x^2 = x^2y^{-3},\ z = y^{-3}\rangle \longrightarrow$
$\langle x,y,z|\ x^2 = yxy,\ y^{-1} = x^2y^{-3}x^{-2}y^3,\ z = y^{-3}\rangle \longrightarrow$
$\langle x,y,z|\ x^2 = yxy,\ y^{-1} = x^2zx^{-2}z^{-1},\ z = (x^2zx^{-2}z^{-1})^3\rangle \longrightarrow$
$\langle x,y,z|\ x^2 = zx^2z^{-1}x^{-1}zx^2z^{-1}x^{-2},\ y = zx^2z^{-1}x^{-2},\ z = [x^2,z]^3\rangle \longrightarrow$
$\langle x,z|\ x^4 = zx^2z^{-1}x^{-1}zx^2z^{-1},\ z = [x^2,z]^3\rangle \longrightarrow (z \to y)$
$\langle x,y|\ x^4 = yx^2y^{-1}x^{-1}yx^2y^{-1},\ y = [x^2,y]^3\rangle$. $\square$

Proof of Proposition 1.3. $\langle x,y|\ x^n = y^{n+1},\ xyx = yxy\rangle \longrightarrow \langle x,y|\ x^n = y^{n+1},\ x = (y^{-1}x^{-1})y(xy)\rangle \longrightarrow$ (substitution) $\langle x,y|\ y^{-1}x^{-1}y^nxy = y^{n+1},\ x = (y^{-1}x^{-1})y(xy)\rangle$(!) $\longrightarrow \langle x,y|\ x^{-1}y^nx = y^{n+1},\ x = y^{-1}x^{-1}yxy\rangle$. $\square$

3. Potential counterexamples from knot diagrams

In this section, we give a rather general and simple method for constructing potential counterexamples to the original Andrews-Curtis conjecture based on "exotic" knot diagrams of the unknot. Note however that it cannot produce counterexamples to the weak Andrews-Curtis conjecture since it is known that Reidemeister moves applied to a knot diagram give stably AC-equivalent Wirtinger presentations of the knot group.

The method can be summarized as follows. Take a piece of rope, glue the ends, crumple (or even tie) the rope, and through it on the table. What you have now on the table, is an "exotic" diagram of the unknot. (We note that you can get a diagram of the unknot by just drawing a closed curve on a list of paper, making each self-crossing an undercrossing until you eventually close the curve, but this method cannot possibly give you a desired counterexample!)

Now comes the mathematical part of the method. If you are lucky, the number of crossings in your knot diagram cannot be reduced by a single Reidemeister move (see e.g. [14, p.9]). If it can, you have to "manually" adjust your diagram until you get one with that property (otherwise, you will have to do extra work in the next stage). Now write down the Wirtinger presentation of the fundamental group of the unknot (which is the infinite cyclic group) based on your knot diagram. This is a balanced presentation, with the number of generators equal to the number of crossings in the knot diagram. (A particularly simple explanation of how to get the Wirtinger presentation from a knot diagram is given on p.268 of the book [14]). If the ambient free group is generated by $x_1, ..., x_n$, then the normal closure of the relators is the same as that of $x_1 x_2^{-1}, ..., x_1 x_n^{-1}$. All Wirtinger relators are of the form $x_i = x_j x_k x_j^{-1}$, or $x_i = x_j^{-1} x_k x_j$. Furthermore, any one of the relators is redundant, i.e., it follows from the other $(n-1)$.

Based on these well-known facts, we see that if we discard one relator from our Wirtinger presentation, and then add a relator which would trivialize the group, we shall get a balanced presentation of the trivial group. We are now ready for

Proof of Theorem 1.4. Consider the following diagram of the unknot (Figure 1).

The corresponding Wirtinger presentation is $\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8,$ $x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14} \mid x_1 = x_{10} x_{14} x_{10}^{-1}$, $x_2 = x_{10}^{-1} x_1 x_{10}$, $x_3 = x_1^{-1} x_2 x_1$, $x_4 = x_6^{-1} x_3 x_6$, $x_5 = x_{12} x_4 x_{12}^{-1}$, $x_6 = x_7^{-1} x_5 x_7$, $x_7 = x_4^{-1} x_6 x_4$, $x_8 = x_1 x_7 x_1^{-1}$, $x_9 = x_{11}^{-1} x_8 x_{11}$, $x_{10} = x_{14} x_9 x_{14}^{-1}$, $x_{11} = x_2^{-1} x_{10} x_2$, $x_{12} = x_1^{-1} x_{11} x_1$, $x_{13} = x_5 x_{12} x_5^{-1}$, $x_{14} = x_1 x_{13} x_1^{-1} \rangle$.

Now we discard the relator $x_{12} = x_1^{-1} x_{11} x_1$, and then eliminate the generators one by one, starting with $x_{14} = x_2$ (from the first two relators). That means, we apply relevant AC-transformations (cf. the "basic substitution principle" in the previous section) to the first two relators to get $x_{14} = x_2$; then replace x_{14} by x_2 in all relators that involve x_{14} (the "basic substitution principle" again), and then get rid of x_{14} by applying the transformation (AC5).

In a similar fashion, we successively eliminate $x_{11} = x_9$, $x_9 = x_8$, x_{13}, x_8, x_4, x_3, x_6, x_{10}, x_2, and x_1. In the end, we get the following presentation: $\langle x_5, x_7, x_{12} \mid x_5 = x_{12} x_7^{-1} x_5^{-1} x_7 x_5 x_{12} x_5^{-1} x_7^{-1} x_5 x_7 x_{12}^{-1}$, $x_7 = x_5^{-1} x_7 x_5 x_{12}^{-1} x_5^{-1} x_7^{-1} x_5 x_7 x_5 x_{12} x_5^{-1} x_7^{-1} x_5 \rangle$.
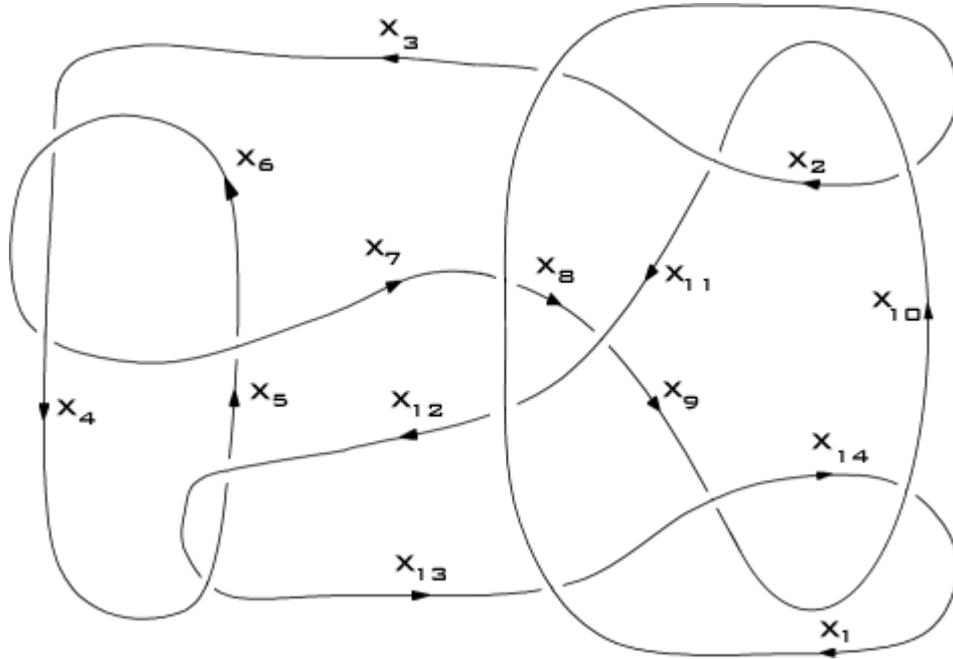
Figure 1: A diagram of the unknot.

Denote $x = x_5$, $y = x_7$, $z = x_{12}$. To make this group trivial, we have to add a relator which would, together with the ones that we have, generate the group $F_3 = \langle x, y, z \rangle$ modulo its commutator subgroup. Since the abelianization of our relators is $\{xz^{-1}, x^{-1}y\}$, adding any relator whose exponent sum on x, y and z equals ± 1, will trivialize the group. This completes the proof. $\square$

Remark. The presentation of the trivial group obtained by this method *very much* depends on the following:

(1) the relator discarded in the very beginning;

(2) the order in which generators are eliminated.

By varying these choices, one can obtain several essentially different balanced presentations of the trivial group from the same knot diagram.

4. The semigroup of balanced presentations of the trivial group

We start with

Lemma 4.1. Let $P = \langle X \mid R \rangle$, $Q = \langle X \mid S \rangle$ be two balanced presentations of the trivial group. Then $P \circ Q = \langle X \mid R(S) \rangle$ also presents the trivial group.

Proof. Since the group defined by P is trivial, every element $x_i \in X$ can

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be expressed as a product of conjugates of relators from R, say

$$x_i = \prod_j r_{ij}(X)^{w_{ij}(X)}.$$

Upon replacing x_i by $s_i(X)$ (i.e., upon applying an endomorphism of $F(X)$ to both sides), we get

$$s_i = \prod_j r_{ij}(S)^{w_{ij}(S)}.$$

Therefore, $S \subset \langle R(S) \rangle^F$, hence $\langle R(S) \rangle^F \supseteq \langle S \rangle^F = F$. This shows that $P \circ Q$ defines the trivial group. $\square$

Proof of Proposition 1.5. It suffices to show that if a balanced presentation $P = \langle X \mid R \rangle$ is AC-equivalent to the standard one, then for any balanced presentation $Q = \langle X \mid S \rangle$ the composition $P \circ Q = \langle X \mid R(S) \rangle$ is AC-equivalent to the Q.

Let $t_1, ..., t_n$ be a sequence of transformations (AC1)-(AC3) that takes P to the standard presentation of the trivial group. Denote by $u_1, ..., u_n$ a chain of Andrews-Curtis transformations obtained from $t_1, ..., t_n$ as follows:
– if t_i is a transformation of the type (AC1) or (AC2), then $u_i = t_i$;
– if t_i is of the type (AC3) (say, t_i replaces r_j by $w(X)r_j w(X)^{-1}$), then u_i replaces r_j by $w(S)r_j w(S)^{-1}$.

Obviously, the sequence $s_1, \ldots, s_n$ transforms $P \circ Q$ into Q. $\square$

Example. Let $P = \langle x, y \mid y^{-1}xy = x^2, x^{-1}yx = y^2 \rangle$. It is easy to see that P is AC-equivalent to the standard presentation $\langle X \mid X \rangle$. If $Q = \langle x, y \mid r, s \rangle$ is an arbitrary presentation of the trivial group, then, by Proposition 1.5, the presentation

$$P \circ Q = \langle x, y \mid r^{-1}sr = s^2, s^{-1}rs = r^2 \rangle$$

is AC-equivalent to Q.

5. The Andrews-Curtis equivalence in metabelian groups

We recall that given a group G presented in the form $G = F/R$, one can turn the abelian group R/R' into a left $\mathbf{Z}G$-module upon setting $g(rR') = g^{-1}rgR'$, $g \in G$, $r \in R$, and extending this action $\mathbf{Z}$-linearly to the whole ring $\mathbf{Z}G$.

One more definition that we need seems to be well-known: a presentation $G = F/R$ is called aspherical if the corresponding relation module R/R' is a free $\mathbf{Z}G$-module. Now we have

Proposition 5.1. Let a group G be given by an aspherical presentation $G = F/R$, and let the corresponding relation module R/R' be generated by k elements. Then: any two generating systems of the relation module R/R' are (stably) AC-equivalent if and only if every matrix from the group $GL_k(\mathbf{Z}G)$ is (stably) tame, and all diagonal matrices in the corresponding

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decompositions have trivial units (i.e., elements of the form $\pm g$, $g \in G$) on the diagonal.

This suggests two ways of constructing counterexamples. The first way is to pick up an aspherical presentation $G = F/R$ of a group G such that the group ring $\mathbf{Z}G$ has non-trivial units; this idea has been essentially used in [15]. The second way, the one that we use here, is to choose a group G such that there are non-tame invertible matrices over the group ring $\mathbf{Z}G$.

Proof of Proposition 5.1. Fix a generating system $Y = \{r_1 R', ..., r_k R'\}$ of the relation module R/R', and let $Y = \{s_1 R', ..., s_k R'\}$ be another generating system. Then we can write:

$$s_i \equiv \prod_{j=1}^{k} r_j^{u_{ij}} \ (mod \ R'), \ 1 \leq i \leq k, \tag{1}$$

where $u_{ij} \in \mathbf{Z}G$. Similarly,

$$r_i \equiv \prod_{j=1}^{k} s_j^{v_{ij}} \ (mod \ R'), \ 1 \leq i \leq k, \tag{2}$$

for some $v_{ij} \in \mathbf{Z}G$. Hence, if we define two $k \times k$ matrices $U = (u_{ij})$ and $V = (v_{ij})$ over the group ring $\mathbf{Z}G$, then (1) and (2) can be written in the matrix form as $(s_1 R', ..., s_k R')^t = U \cdot (r_1 R', ..., r_k R')^t$, and $(r R', ..., r R')^t = V \cdot (s R', ..., s R')^t$, where ^{t} means transposition, i.e., we consider our k-tuple as a column. The multiplication by elements of $\mathbf{Z}G$ is interpreted here as the module action described above.

This yields $VU \cdot (r_1 R', ..., r_k R')^t = (r_1 R', ..., r_k R')^t$. Since R/R' is a free $\mathbf{Z}G$-module and $\{r_1 R', ..., r_k R'\}$ a free basis, this is equivalent to $VU = I$, the identity matrix, therefore

$$U, V \in GL_k(\mathbf{Z}G). \tag{3}$$

We now observe that applying an elementary transformation (AC1) to a k-tuple
$(r_1 R', ..., r_k R')$ amounts to multiplying the corresponding column by an elementary matrix on the left, and applying (AC2) or (AC3) amounts to multiplying it by a diagonal matrix which has only trivial units on the diagonal. Conversely, a multiplication by any such a diagonal or elementary matrix on the left can be interpreted as a (sequence of) elementary transformation(s) (AC1)–(AC3). Combining this with (3) yields the result. □

Proof of Theorem 1.6 follows almost immediately from Proposition 5.1. The group G in this situation is the free abelian group of rank 2 generated by the natural images of x_1 and x_2; the presentation $< x_1, x_2, x_3 | \ [x_1, x_2], x_3 >$ is obviously aspherical. The group ring $\mathbf{Z}G$ is the ring of Laurent polynomials in 2 variables. Over this ring, there are non-tame 2×2 invertible matrices

– see [3]. Moreover, the index of the subgroup $GE_2(\mathbf{Z}G)$ generated by elementary and diagonal matrices, in the whole group $GL_2(\mathbf{Z}G)$, is infinite [3]. Hence, the first part of Theorem 1.4 follows now from Proposition 5.1. The particular pair of elements in the statement of Theorem 1.6 is modeled on the example of a matrix in $GL_2(\mathbf{Z}G)$, but not in $GE_2(\mathbf{Z}G)$, given in [9]. The matrix is:

$$\begin{pmatrix} 1 - 2(x_1 - 1)x_2^{-1} & 4x_2^{-1} \\ -(x_1 - 1)^2 x_2^{-1} & 1 + 2(x_1 - 1)x_2^{-1} \end{pmatrix}$$

On the other hand, by a result of Suslin [25], $GE_k(\mathbf{Z}G) = GL_k(\mathbf{Z}G)$ for any $k \geq 3$; in particular, every matrix from $GL_2(\mathbf{Z}G)$ is stably tame, whence the result about stable AC-equivalence. $\square$

Proof of Proposition 1.7. In this situation, the group G is infinite cyclic (generated by x_3), so that the group ring $\mathbf{Z}G$ is the ring of one-variable Laurent polynomials. The result now follows from Proposition 5.1. $\square$

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