On isomorphism of Lie algebras with one defining relation

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Abstract

Let L be a finitely generated free Lie algebra. We construct an example of two elements u and v of L such that the factor algebras $L/(u)$ and $L/(v)$ are isomorphic, where (u) and (v) are ideals of L generated by u and v, respectively, but there is no automorphism φ of L such that $\varphi(u) = v$.

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1 Introduction

Let L be a finitely generated free Lie algebra. The main result of this paper is an example of two elements u and v of L such that the factor algebras $L/(u)$ and $L/(v)$ are isomorphic, where (u) and (v) are ideals of L generated by u and v, respectively, but there is no automorphism φ of L such that $\varphi(u) = v$.

Answering a question of Magnus, McCool and Pietrowski [6] have constructed examples of this kind for one-relator groups. Later Brunner [1] produced an infinite series of isomorphic one-relator groups with pairwise inequivalent relators. Earlier, Magnus [5], [4, Proposition 5.10] proved that if w is an element of a finitely generated free group F such that the factor group of F by the normal subgroup generated by w is a free group, then w is a primitive element of the group F, i.e., a member of a free generating set of F. Similar result for free Lie algebras was obtained by Kukin [3]. Shpilrain and Yu [10] showed that similar result holds for free associative algebras of rank 2, but does not hold for free associative algebras of rank ≥ 3. For the rank 2 case, see also [2].

We introduce here three types of isomorphism-preserving "elementary" transformations that can be applied to an arbitrary factor algebra $L(x_1, \ldots, x_n)/R$, where $L = L(x_1, \ldots, x_n)$ is a free Lie algebra with free generators $x_1, \ldots, x_n$, and R an ideal of L. These transformations are modeled on Tietze transformations in groups (see e.g. [4]).

(1) *Introducing a new variable*:
replace $L(x_1, \ldots, x_n)/R$ by $L(x_1, \ldots, x_n, y)/R'$, where $R' = R + (w)$, (w) is the ideal generated by $w = y - f(x_1, \ldots, x_n)$, and $f(x_1, \ldots, x_n)$ is an arbitrary element of L.

(2) *Canceling a variable*:
if we have an algebra of the form $L(x_1, \ldots, x_n, y)/(f_1, \ldots, f_m, g)$, where g is of the form $y - f(x_1, \ldots, x_n)$, and $f_1, \ldots, f_m \in L(x_1, \ldots, x_n)$, then replace this algebra by $L(x_1, \ldots, x_n)/(f_1, \ldots, f_m)$.

(3) *Renaming the variables*:
replace variables $x_1, \ldots, x_n$ by $x_{i_1}, \ldots, x_{i_n}$, where $i_1, \ldots, i_n$ are arbitrary distinct indices, not necessarily the integers in $\{1, \ldots, n\}$.

The following theorem can be proved along the same lines as the original Tietze theorem (see [4]).

Theorem 1 *Two algebras*

$$L(x_1, \ldots, x_n)/(f_1, \ldots, f_m) \text{ and } L(x_1, \ldots, x_n)/(h_1, \ldots, h_k)$$

are isomorphic if and only if one can get from one of them to the other by a sequence of transformations (1)–(3).

This theorem also has the following interesting corollary which says that, although for isomorphic algebras

$$L(x_1, \ldots, x_n)/(f_1, \ldots, f_m) \text{ and } L(x_1, \ldots, x_n)/(g_1, \ldots, g_k),$$

the ideals $(f_1, \ldots, f_m)$ and $(g_1, \ldots, g_k)$ do not have to be equivalent under an automorphism of $L(x_1, \ldots, x_n)$, they are always *stably equivalent* (the precise meaning of that is clear from the statement below). This complements our Corollary 5 in the next section nicely.

Corollary 2 *If two algebras*

$$L(x_1, \ldots, x_n)/(f_1, \ldots, f_m) \text{ and } L(x_1, \ldots, x_n)/(g_1, \ldots, g_k)$$

are isomorphic, then the ideals

$$(f_1, \ldots, f_m, x_{n+1}, \ldots, x_{2n}) \text{ and } (g_1, \ldots, g_k, x_{n+1}, \ldots, x_{2n})$$

are equivalent under an automorphism of $L(x_1, \ldots, x_{2n})$.

2 Main result

Theorem 3 *Let* K *be a field,* $L = L(x, y, z)$ *the free Lie algebra over* K *with the set* $\{x, y, z\}$ *of free generators. Let*

$$u = x - [[x, y], [y, z]], \ v = x + [[[y, z], x], y],$$

and let (u) *and* (v) *be the ideals of* L *generated by the elements* u *and* v*, respectively. Then* $L/(u) \cong L/(v)$*, but there is no automorphism* φ *of* L *such that* $\varphi(u) = v$.

Proof. Using "Tietze transformations" (1)–(3), we get

$$\begin{aligned} L/(u) = \ & L(x, y, z) \ / \ (x - [[x, y], [y, z]]) \cong \\ & L(x, y, z, t) \ / \ (t - [x, y], x - [[x, y], [y, z]]) \cong \\ & L(x, y, z, t) \ / \ (t - [x, y], x - [t, [y, z]]) \cong \\ & L(y, z, t) \ / \ (t - [[t, [y, z]], y]) \cong \\ & L(x, y, z) \ / \ (x + [[[y, z], x], y]) = L/(v). \end{aligned}$$

Let d be the standard degree function in the free Lie algebra L ($\mathrm{d}\,(x) = \mathrm{d}\,(y) = \mathrm{d}\,(z) = 1$, $\mathrm{d}\,(u) = \mathrm{d}\,(v) = 4$). By $\widetilde{f}$ we denote the leading homogeneous component of an element f of L.

If φ is an automorphism of L, then the leading component $\widetilde{\varphi(u)}$ belongs to the second derived subalgebra $L^{(2)} = [[L, L], [L, L]]$ of the algebra L (it is obvious that $\widetilde{v} \notin L^{(2)}$, and therefore $v \neq \varphi(u)$). Indeed, let $\varphi(x) = a$, $\varphi(y) = b$, $\varphi(z) = c$.

If the leading components $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ are linearly independent, then it is clear that

$$\widetilde{\varphi(u)} = [[\widetilde{b}, \widetilde{c}], [\widetilde{a}, \widetilde{b}]], \ \mathrm{d}\,(\varphi(u)) > \mathrm{d}\,(\varphi(x)) = \mathrm{d}\,(\widetilde{a}) = \mathrm{d}\,(a).$$

Suppose now that the leading components $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ are linearly dependent. If $\widetilde{a}$ and $\widetilde{b}$ are linearly independent, then $\mathrm{d}\,([a, b]) = \mathrm{d}\,([\widetilde{a}, \widetilde{b}]) > \mathrm{d}\,(a)$. If $\widetilde{c} = \alpha\widetilde{a} + \beta\widetilde{b}$, where $\alpha, \beta \in K$, then we consider the element $c' = c - \alpha a - \beta b$. We have

$$\widetilde{[[b, c], [a, b]]} = [[\widetilde{b}, \widetilde{c'}], [\widetilde{a}, \widetilde{b}], \ \mathrm{d}\,(\varphi(u)) = \mathrm{d}\,([[\widetilde{b}, \widetilde{c'}], [\widetilde{a}, \widetilde{b}]) > \mathrm{d}\,(a).$$

If $\widetilde{a} = \alpha\widetilde{b}$, where $\alpha \in K$, then $\mathrm{d}\,(a) = \mathrm{d}\,(b)$. Consider the element $a' = a - \alpha b$. Since the leading components $\widetilde{a'}$ and $\widetilde{b}$ are linearly independent, as before we get

$$\widetilde{\varphi(u)} = [[\widetilde{b}, \widetilde{c}], [\widetilde{a'}, \widetilde{b}]], \ \mathrm{d}\,(\varphi(u)) > \mathrm{d}\,(a).$$

If $\widetilde{c} = \alpha\widetilde{a'} + \beta\widetilde{b}$, where $\alpha, \beta \in K$, then

$$\widetilde{\varphi(u)} = [[\widetilde{b}, \widetilde{c'}], [\widetilde{a'}, \widetilde{b}]], \ \text{where } \ c' = c - \alpha a' - \beta b,$$

and again $\mathrm{d}\,(\widetilde{\varphi(u)}) > \mathrm{d}\,(a) = \mathrm{d}\,(b)$.

Note that the elements a' and c' constructed above are nonzero becase φ is an automorphism of the algebra L.

Thus, we showed that $\widetilde{\varphi(u)} \in L^{(2)}$ for any automorphism of L. This completes the proof. ■

We note that the main difficulty in the proof of Theorem 3 was establishing the inequivalence of two given elements under an automorphism of L because there is no known algorithm for detecting such inequivalence. However, in a special case of the two-generated free Lie algebra, the situation is easier because all automorphisms of the free Lie algebra of rank 2 are linear.

Proposition 4 *Let K be a field, $L = L(x, y)$ the free Lie algebra over K with the set $\{x, y\}$ of free generators. Let*

$$u = x - [[[x, y], y], [x, y]], \ v = x - [[[x, y], x], y],$$

and let (u) and (v) be the ideals of L generated by the elements u and v, respectively. Then $L/(u) \cong L/(v)$, but there is no automorphism φ of L such that $\varphi(u) = v$.

Proof. By using Tietze transformations, we get

$$\begin{aligned}
L/(u) = \; & L(x,y) & / \; & (x - [[[x,y],y],[x,y]]) \cong \\
& L(x,y,t) & / \; & (x - [[[x,y],y],[x,y]], t - [x,y]) \cong \\
& L(x,y,t) & / \; & (x - [[t,y],t], t - [[[t,y],t],y] \cong \\
& L(y,t) & / \; & (t - [[[t,y],t],y]) \cong \\
& L(x,y) & / \; & (x - [[[x,y],x],y]) = L/(v).
\end{aligned}$$

It is well known that all automorphisms of the two-generated free Lie algebra are linear. Therefore, for any automorphism φ of L we have $\mathrm{d}\,(\varphi(u)) = \mathrm{d}\,(u) = 5 \neq 4 = \mathrm{d}\,(v)$. This completes the proof. ■

Corollary 5 *There are isomorphic one-relator Lie algebras $L/(u)$ and $L/(v)$ such that the isomorphism is not induced by any automorphism of the ambient free Lie algebra L. Or, equivalently, there is no automorphism of L that takes the ideal (u) to (v).*

Proof. We take the same L, u and v as in Theorem 3, or as in Proposition 4. Shirshov [9] showed that if in a free Lie algebra the ideals (a) and (b) generated by nonzero elements a and b coincide, then $b = \alpha a$, $\alpha \in K$. Combining this fact with our Theorem 3 yields the result. ■

3 An open problem

Problem *Let* char $K = p > 2$. *Let L^p be a finitely generated free Lie p-algebra, and w a nonzero element of L^p such that the factor algebra $L^p/(w)$ is a free Lie p-algebra. Is it true that w is a primitive element of L^p?*

In his proof of a similar result for free Lie algebras, Kukin [3] essentially used the Freiheitssatz for Lie algebras which is due to Shirshov [9]. It is not known whether the Freiheitssatz holds for free Lie p-algebras. Actually, for free algebras of homogeneous Schreier varieties of algebras in which the Freiheitssatz holds (in particular, for the varieties of all algebras, of all commutative algebras, of all anticommutative algebras, of Lie algebras) this problem has a positive solution, see [7, Proposition 2, Theorem 5]. Originally, the Freiheitssatz in group theory is due to Magnus [5].

We also note that, by using free differential calculus, A. A. Mikhalev and Zolotykh [8] constructed fast matrix algorithms for recognizing primitive elements of free Lie (p-)(super)algebras. Similar algorithms for free non-associative algebras were constructed by A. A. Mikhalev, Umirbaev and Yu [7].

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References

[1] A. M. Brunner, *A group with an infinite number of Nielsen inequivalent one-relator presentations*. J. Algebra **42** (1976), 81–84.

[2] V. Drensky and J.-T. Yu, *Primitive elements of free metabelian algebras of rank two*, Internat. J. Algebra and Comput., to appear.

[3] G. P. Kukin, *Primitive elements of free Lie algebras*. Algebra i Logika **9** (1970), no. 4, 458–472. English translation: Algebra and Logic **9** (1970), 275–284.

[4] R. Lyndon and P. Schupp, *Combinatorial Group Theory*. Reprint of the 1977 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[5] W. Magnus, *Über diskontinuierliche Gruppen mit einer definierden Relation (Der Freiheitssatz)*. J. Reine Angew. Math. **163** (1930), 141–165.

[6] J. McCool and A. Pietrowski, *On free products with amalgamation of two infinite cyclic groups*. J. Algebra **18** (1971), 377–383.

[7] A. A. Mikhalev, U. U. Umirbaev, and J.-T. Yu, *Automorphic orbits in free non-associative algebras*. J. Algebra **243** (2001), 198–223.

[8] A. A. Mikhalev and A. A. Zolotykh, *Rank and primitivity of elements of free color Lie (p-)superalgebras*. Intern. J. Algebra and Computation **4** (1994), 617–656.

[9] A. I. Shirshov, *Some algorithmical problems for Lie algebras*. Sib. Mat. Zh. **3** (1962), 292–296.

[10] V. Shpilrain and J.-T. Yu, *Factor algebras of free algebras: on a problem of G. Bergman*. Bull. London Math. Soc., to appear.