Counting primitive elements of a free group

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Abstract. We show that the number of primitive elements of length $n$ in a free group of rank $r \geq 3$ is bounded from above by $c \cdot \lambda^n$, where $\lambda$ is the greatest real root of the polynomial $\lambda(\lambda^2 - 1)(\lambda - (2r - 2)) + 1$ and $c$ is some constant. To obtain this estimate, we use some well known facts from graph theory together with a counting technique based on the theory of linear recurrence relations.

1 Introduction

Let $F_r$ be the free group of a finite rank $r \geq 2$ with a set $X = \{x_i\}, 1 \leq i \leq r$, of free generators. An element $g \in F_r$ is called primitive if it is a member of some free basis of $F_r$. Or, equivalently, if there is an automorphism of $F_r$ that takes $g$ to $x_1$.

Let $P(r, n)$ be the number of primitive elements of length $n$ in $F_r$. This paper addresses the following problem of M. Wicks (see [1, Problem (F17)]):

**Problem.** What is the growth of $P(r, n)$ as a function of $n$, with $r$ fixed?

For $r = 2$, the number $P(2, n)$ grows like $(\sqrt{3})^n$; this follows from [3, Proposition 4.1], as observed by Rivin [6]. We also note that the precise number of cyclically reduced primitive elements of length $n$ in $F_2$ was given in [5]; this number has linear growth as a function of $n$. This case however is exceptional, and it is easy to see that if $r \geq 3$, then the growth of the number of primitive elements (cyclically reduced or not) is exponential, say, $\lambda^n$. The problem is therefore to determine $\lambda$ (as a function of $r$).

It was shown in [3] that $\lambda \leq \mu(r) > 2r - 2$, where $\mu(r)$ is asymptotically approaching $2r - 2$. In [2], it was shown that $\lambda \leq 2r - 2$. In both papers, crucial for the counting arguments was the fact that the Whitehead graph

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of any primitive element of length \( > 2 \) has either an isolated edge or a cut vertex, i.e., a vertex that, having been removed from the graph together with all incident edges, increases the number of connected components of the graph.

In the present paper, we again exploit this fact to sharpen the estimate for \( \lambda \):

**Theorem.** If \( r \geq 3 \), then, for some constants \( c_1, c_2 \), one has
\[
c_1 \cdot n \cdot (2r - 3)^n \leq P(r, n) \leq c_2 \cdot \lambda^n,
\]
where \( \lambda \) is the greatest real root of the polynomial
\[
\lambda(\lambda^2 - 1)(\lambda - (2r - 2)) + 1.
\]

Obviously, \( \lambda \) in the Theorem is smaller than \( 2r - 2 \), although it is asymptotically approaching \( 2r - 2 \). It seems that the above estimate exhausts the potential of the “cut vertex” approach, and further improvements would require brand new ideas. As for the ultimate result, we make the following

**Conjecture.** If \( r \geq 3 \), then \( P(r, n) = O(n \cdot (2r - 3)^n) \).

We note that \( n \cdot (2r - 3)^n \) is (up to a factor that does not depend on \( n \)) the number of “obvious” primitive elements of length \( n \) having the form \( u \cdot x_i^{\pm 1} \cdot v \), where \( u, v \) are arbitrary elements that do not depend on \( x_i \). Thus, our conjecture says that, for \( r \geq 3 \), “most” primitive elements are of that form.

Our proof uses a refined analysis of the Whitehead graph of a primitive element and a new ingredient, namely, systems of linear recurrence relations, which is described in the following section. We believe this technique can be used for other counting problems in (free) groups as well.

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## 2 The Whitehead graph, Whitehead tree and transition matrix

The Whitehead graph \( Wh(u) \) of a (cyclically reduced) word \( u \in F_r \) is obtained as follows. The vertices of this graph correspond to the elements of the free generating set \( X \) and their inverses. For each occurrence of a subword \( x_i x_j \) in the word \( u \), there is an edge in \( Wh(u) \) that connects the vertex \( x_i \) to the vertex \( x_j^{-1} \); if \( u \) has a subword \( x_i x_j^{-1} \), then there is an edge connecting \( x_i \) to \( x_j \), etc. There is one more edge (the external edge) included in the definition of the Whitehead graph: this is the edge that connects the vertex corresponding to the last letter of \( u \) to the vertex corresponding to the inverse of the first letter.

It was observed by Whitehead himself that the Whitehead graph of any primitive element of length \( > 2 \) has either an isolated edge or a cut vertex,
i.e., a vertex that, having been removed from the graph together with all incident edges, increases the number of connected components of the graph.

Both [2] and [3] exploited this fact to get upper bounds for $P(r,n)$. In the present paper, we refine these upper bounds based on the following simple observation (see e.g. [7, Exercise 5.7]):

Suppose a simple graph $\Gamma$ has the property that any two vertices can be included in a simple circuit. Then $\Gamma$ does not have a cut vertex.

Our strategy therefore will be to produce an upper bound for the number of elements $u \in F_r$ whose Whitehead graph (or, rather, the underlying simple graph) does not have the property alluded to in the previous observation.

Our counting technique is best visualized by introducing the Whitehead tree $WT(u)$ associated to the graph $Wh(u)$. The Whitehead tree $WT(u)$ is an infinite rooted labeled tree; the root (level 0) is labeled 1; 2r vertices at level 1 are labeled by $x_i$ or $x_i^{-1}$, $1 \leq i \leq r$, and a parent vertex at a level $k \geq 1$ labeled $x_i$ is connected to a child vertex at the level $(k+1)$ labeled $x_j$ if and only if there is an edge from the vertex $x_j$ to the vertex $x_j^{-1}$ in the graph $Wh(u)$ (or, equivalently, if $u$ has a subword $x_i x_j$). Thus, by computing the growth function of the tree $WT(u)$ (i.e., by counting elements at a level $n$ as a function of $n$), we find (or estimate) the number of words $u$ of length $n$ having the Whitehead graph with a particular underlying simple graph.

Now the problem of finding the growth function of the tree $WT(u)$ can be translated into a system of (linear) recurrence relations as follows. Let $j_n$ (resp. $\bar{j}_n$) be the number of vertices labeled $x_j$ (resp. $x_j^{-1}$) at the level $n$ of the tree $WT(u)$. Then, based on the Whitehead graph $Wh(u)$, it is easy to write down recurrence relations for all $j_{n+1}$ and $\bar{j}_{n+1}$ in terms of appropriate $i_n$ and $\bar{i}_n$. All these recurrence relations are obviously linear, so one can employ a well studied theory of systems of linear recurrence relations to solve for all $j_n$ and $\bar{j}_n$.

Solving a system like that basically amounts to finding eigenvalues of the matrix of this system, usually called the transition matrix. (Note that all entries of this matrix are nonnegative.) In fact, since we are only interested in the growth of $\sum_n (j_n + \bar{j}_n)$, we only need the maximum real positive eigenvalue (the Perron-Frobenius eigenvalue) of the transition matrix; this will determine the growth type of the tree $WT(u)$.

Of course, a system like that is, in general, too big; it has $2r$ equations. In most practical situations however the set of vertices of the Whitehead graph $Wh(u)$ can be split into a union of a few (large) sets, and then one can write recurrence relations (still linear) for the numbers of elements in these sets, thus getting a reasonable number of equations. We illustrate this in the following section.

Finally, we mention that some sets of elements of $F_r$ can be interpreted as Markov processes whose transition matrices arise in a way similar to
what we described above (we refer to [4] for details). It is conceivable that
different sets like that are precisely graphic sets in the sense of [3].

3 Proof of the main result

Let $u \in F_r$, and let $Wh(u)$ be the underlying simple graph of the Whitehead
graph $Wh(u)$ (in particular, $Wh(u)$ is not oriented and has no loops or
multiple edges).

There are two principal cases to consider:

Case 1. There is a vertex of degree 1 in $Wh(u)$.

Case 2. The degree of any vertex of the graph $Wh(u)$ is $\geq 2$.

In Case 1, suppose that, say, the vertex of $Wh(u)$ corresponding to the
generator $x_1$ has degree 1. That means $u$, considered as a cyclic word,
has the form $u = u(x_1x_jx_2, ..., x_r)$ for some $j \neq 1$. Then the vertex
corresponding to $x_j^{-1}$ may have any degree $\leq 2r - 1$, whereas other vertices
have degrees $\leq 2r - 2$.

Thus, the greatest possible number of edges in the graph $Wh(u)$ occurs
if $Wh(u)$ has one vertex of degree 1, one vertex of degree $2r - 1$, and
$2r - 2$ vertices of degree $2r - 2$. To produce the corresponding transition
matrix, we let $a_n$ be the number of vertices labeled $x_1$ in the $n$th level
of the Whitehead tree $WT(u)$, $b_n$ the number of vertices labeled $x_j^{-1}$, $c_n$
the number of vertices labeled $x_j$, and $r_n$ the number of vertices labeled
by other letters. Then we have the following system of linear recurrence relations:

$$
\begin{align*}
\begin{cases}
a_{n+1} &= b_n + c_n + r_n \\
b_{n+1} &= b_n + r_n \\
c_{n+1} &= a_n + c_n + r_n \\
r_{n+1} &= b_n \cdot (2r - 3) + c_n \cdot (2r - 4) + r_n \cdot (2r - 4)
\end{cases}
\end{align*}
$$

(1)

Here we are assuming that $n \geq 1$ and use the initial conditions $a_1 =
b_1 = c_1 = 1$, and $r_1 = 2r - 3$.

To explain, say, the last recurrence relation, we see that, given the
assumptions of Case 1, $u$ cannot have a subword $x_jx_j^{-1}$. Then, since $r_{n+1}$
is the number of vertices labeled by letters other than $x_1$, $x_j$, $x_j^{-1}$, we
see that a vertex at the level $n$ labeled by $x_j$ may have (at most) $2r - 4$
children at the level $(n + 1)$, which is why $c_n$ is multiplied by $2r - 4$ in the
last recurrence relation.

We also note that the actual value of $r_{n+1}$ is smaller than $b_n \cdot (2r -
3) + c_n \cdot (2r - 4) + r_n \cdot (2r - 4)$ because, say, some vertices labeled $x_j$
may be connected to vertices labeled $x_j$, and not to vertices labeled by “other”

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letters. However, since we are looking for an upper bound, we are allowed to replace the actual value of \( r_{n+1} \) by a greater one.

The transition matrix of the system (1) is

\[
\begin{pmatrix}
0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 2r - 3 & 2r - 4 & 2r - 4
\end{pmatrix}.
\]

The characteristic polynomial of this matrix is \( \lambda(\lambda^2 - 1)(\lambda - (2r - 2)) + 1 \), which completes the proof in Case 1.

In Case 2, we start with a simple

**Lemma 3.1.** Let \( r \geq 2 \). Suppose the degree of any vertex of a simple graph \( \Gamma \) with \( 2r \) vertices is \( \geq 2 \). If \( \Gamma \) has either at least two vertices of degree \( 2r - 2 \) or at least one vertex of degree \( 2r - 1 \), then any two vertices of \( \Gamma \) can be included in a simple circuit.

**Proof.** Let \( v_1 \) and \( v_2 \) be two vertices of degree \( 2r - 2 \). If they are not adjacent, then they both must be adjacent to all other vertices of the graph \( \Gamma \). Then, given any two vertices \( w_1 \) and \( w_2 \) of \( \Gamma \), we can include them in the simple circuit \( w_1 \rightarrow v_2 \rightarrow w_2 \rightarrow v_1 \rightarrow w_1 \).

If \( v_1 \) and \( v_2 \) are adjacent, then there might be a vertex \( w \) which is not adjacent to either \( v_1 \) or \( v_2 \) (or both). Suppose \( w_1 \) and \( w_2 \) are two vertices of \( \Gamma \). If they both are different from \( w \), then they obviously can be included in a simple circuit. Now suppose, say, \( w_1 = w \). Consider two cases:

(a) \( w \) is adjacent to either \( v_1 \) or \( v_2 \); without loss of generality assume that \( w \) is adjacent to \( v_1 \). Since the degree of any vertex of \( \Gamma \) is \( \geq 2 \), \( w \) must be adjacent to at least one other vertex. If \( w \) is adjacent to \( w_2 \), we get a simple circuit \( w \rightarrow w_2 \rightarrow v_2 \rightarrow v_1 \rightarrow w \). If \( w \) is adjacent to a vertex \( w_3 \neq w_2 \), we get a simple circuit \( w \rightarrow w_3 \rightarrow v_2 \rightarrow w_2 \rightarrow v_1 \rightarrow w \).

(b) \( w \) is not adjacent to either \( v_1 \) or \( v_2 \). Since the degree of any vertex of \( \Gamma \) is \( \geq 2 \), \( w \) must be adjacent to at least two vertices \( w_3 \) and \( w_4 \). The case where \( w \) is adjacent to \( w_2 \) is easy (cf. (a) above), so suppose \( w_3 \neq w_2 \), \( w_4 \neq w_2 \). Then we get a simple circuit \( w \rightarrow w_4 \rightarrow v_1 \rightarrow w_2 \rightarrow v_2 \rightarrow v_3 \rightarrow w \).

The case where \( \Gamma \) has a vertex of degree \( 2r - 1 \) is similar. \( \square \)

From Lemma 3.1, we see that if \( u \in \mathbb{F}_r \) is a primitive element, then at most one vertex of the graph \( Wh(u) \) can have degree \( 2r - 2 \), and there are no vertices of degree \( 2r - 1 \). If there are no vertices of degree \( 2r - 2 \), then the number of elements of length \( n \) with the graph \( Wh(u) \) like that is obviously bounded by \( (2r - 3)^n \), so there is nothing to prove.

Thus, the greatest possible number of edges in \( Wh(u) \) occurs if \( Wh(u) \) has one vertex of degree \( 2r - 2 \), and \( 2r - 1 \) vertices of degree \( 2r - 3 \). We note in passing that, in fact, this cannot happen because, if all vertices of
\(Wh(u)\) have degrees \(\geq 2r - 3\), then, since \(2r - 3 \geq \frac{2r}{2}\) when \(r \geq 3\), \(Wh(u)\) must be Hamiltonian by a classical result of Ore (see e.g. [7, Theorem 7.1, Corollary 7.2]). A Hamiltonian graph cannot have a cut vertex. However, since we are looking for an upper bound, this is legitimate.

Let now the vertex of degree \(2r - 2\) have label \(x_1\). It is easy to see (by induction on the level number) that, for any element \(u\) whose graph \(Wh(u)\) has properties mentioned above, at every level of the Whitehead tree \(WT(u)\) one has at least \(2r - 4\) times as many vertices labeled \(x_i, i \neq 1\), as one does vertices labeled \(x_1\). We therefore have the following inequalities for the numbers \(r_n\) of vertices at level \(n\):

\[
\begin{align*}
r_{n+1} &\leq \frac{1}{2r-3} \cdot r_n \cdot (2r-2) + \frac{2r-4}{2r-3} \cdot r_n \cdot (2r-3) = r_n \cdot (2r-3 + \frac{1}{2r-3}).
\end{align*}
\]

Thus, \(r_n \leq c \cdot (2r-3 + \frac{1}{2r-3})^n\) for some constant \(c\) in this case. Since, if \(r \geq 3\), \(2r - 3 + \frac{1}{2r-3}\) is smaller than the greatest real root of \(\lambda ^2 - 1)(\lambda - (2r-2)) + 1\), this completes the proof in Case 2.

Finally, we note that in the course of our proof we have actually counted cyclically reduced primitive elements of \(F_r\). The result however holds for the number \(P(r,n)\) of all primitive elements as well because for \(r \geq 3\), the number of cyclically reducible primitive elements of a given length \(n\) is easily seen to be smaller than the number of cyclically reduced primitive elements of the same length (cf. [5]). This completes the proof of the theorem. □

References


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