

# ON TWO-GENERATOR SUBGROUPS IN $SL_2(\mathbb{Z})$ , $SL_2(\mathbb{Q})$ , AND $SL_2(\mathbb{R})$

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ABSTRACT. We consider what some authors call “parabolic Möbius subgroups” of matrices over  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$  and focus on the membership problem in these subgroups and complexity of relevant algorithms.

## 1. INTRODUCTION: TWO THEOREMS OF SANOV

Denote  $A(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ ,  $B(k) = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ . In an old paper [11], I. N. Sanov proved two simple yet remarkable theorems:

**Theorem 1.** *The subgroup of  $SL_2(\mathbb{Z})$  generated by  $A(2)$  and  $B(2)$  is free.*

**Theorem 2.** *The subgroup of  $SL_2(\mathbb{Z})$  generated by  $A(2)$  and  $B(2)$  consists of all matrices of the form  $\begin{pmatrix} 1+4n_1 & 2n_2 \\ 2n_3 & 1+4n_4 \end{pmatrix}$  with determinant 1, where all  $n_i$  are arbitrary integers.*

These two theorems together yield yet another proof of the fact that the group  $SL_2(\mathbb{Z})$  is virtually free. This is because the group of all invertible matrices of the form  $\begin{pmatrix} 1+4n_1 & 2n_2 \\ 2n_3 & 1+4n_4 \end{pmatrix}$  obviously has finite index in  $SL_2(\mathbb{Z})$ . Thus, we have:

**Corollary 1.** *The group  $SL_2(\mathbb{Z})$  is virtually free.*

There is another interesting corollary of Theorem 2:

**Corollary 2.** *The membership problem in the subgroup of  $SL_2(\mathbb{Z})$  generated by  $A(2)$  and  $B(2)$  is solvable in constant time.*

We note that this is, to the best of our knowledge, the only example of a natural (and nontrivial) algorithmic problem in group theory solvable in constant time. In fact, even problems solvable in sublinear time are very rare, see [13], and in those that are, one can typically get either “yes” or “no” answer in sublinear time, but not both. Complexity of an input in our case is the “size” of a given matrix, i.e., the sum of the absolute values of its entries. In light of Theorem 2, deciding whether or not a given matrix from  $SL_2(\mathbb{Z})$  belongs to the subgroup generated by  $A(2)$  and  $B(2)$  boils down to looking at residues modulo 2 or 4 of the entries. The latter is decided by looking just at the last one or two digits of each entry (assuming that the entries are given in the binary or, say, decimal form). We emphasize though

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that solving this membership problem in constant time is only possible if an input matrix is known to belong to  $SL_2(\mathbb{Z})$ ; otherwise one would have to check that the determinant of a given matrix is equal to 1, which cannot be done in constant time, although can still be done in sublinear time with respect to the complexity  $|M|$  of an input matrix  $M$ , as defined in the next Section 2, see Corollary 3.

## 2. OUR RESULTS

In this paper, we show that what would be a natural generalization of Sanov's Theorem 2 to  $A(k)$  and  $B(k)$ ,  $k \in \mathbb{Z}_+$ , is not valid for  $k \geq 3$  and moreover, the subgroup generated by  $A(k)$  and  $B(k)$  has infinite index in  $SL_2(\mathbb{Z})$  if  $k \geq 3$ .

**Theorem 3.** *The subgroup of  $SL_2(\mathbb{Z})$  generated by  $A(k)$  and  $B(k)$ ,  $k \in \mathbb{Z}$ ,  $k \geq 3$ , has infinite index in the group of all matrices of the form  $\begin{pmatrix} 1+k^2m_1 & km_2 \\ km_3 & 1+k^2m_4 \end{pmatrix}$  with determinant 1.*

The group of all matrices of the above form, on the other hand, obviously has finite index in  $SL_2(\mathbb{Z})$ .

Our main technical result, proved in Section 3, is the following

**Theorem 4.** *Let  $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$  be a matrix from  $SL_2(\mathbb{R})$ . Call "elementary operations" on  $M$  the following 8 operations: multiplication of  $M$  by either  $A(k)^{\pm 1}$  or by  $B(k)^{\pm 1}$ , on the right or on the left.*

**(a)** *If  $k \in \mathbb{Z}$  and  $M$  belongs to the subgroup of  $SL_2(\mathbb{Z})$  generated by  $A(k)$  and  $B(k)$ , then it has the form  $\begin{pmatrix} 1+k^2n_1 & kn_2 \\ kn_3 & 1+k^2n_4 \end{pmatrix}$  for some integers  $n_i$ .*

*If  $k \in \mathbb{R}$  and  $M$  belongs to the subgroup of  $SL_2(\mathbb{R})$  generated by  $A(k)$  and  $B(k)$ , then it has the form  $\begin{pmatrix} 1+\sum_i k^i n_i & \sum_j k^j n_j \\ \sum_r k^r n_r & 1+\sum_s k^s n_s \end{pmatrix}$  where all  $n_i$  are integers and all exponents on  $k$  are positive integers.*

**(b)** *Let  $k \in \mathbb{R}$ ,  $k \geq 2$ . If  $M \in SL_2(\mathbb{R})$  and there is a sequence of elementary operations that reduces  $\sum_{i,j} |m_{ij}|$ , then there is a single elementary operation that reduces  $\sum_{i,j} |m_{ij}|$ .*

**(c)** *Let  $k \in \mathbb{Z}$ ,  $k \geq 2$ . If  $M \in SL_2(\mathbb{Z})$  and no single elementary operation reduces  $\sum_{i,j} |m_{ij}|$ , then either  $M$  is the identity matrix or  $M$  does not belong to the subgroup generated by  $A(k)$  and  $B(k)$ .*

We also point out a result, similar to Theorem 4, about the *monoid* generated by  $A(k)$  and  $B(k)$  for  $k > 0$ . Unlike Theorem 4 itself, this result is trivial.

**Proposition 1.** *Let  $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$  be a matrix from  $SL_2(\mathbb{R})$ . Call "elementary operations" on  $M$  the following 4 operations: multiplication of  $M$  by either  $A(k)^{-1}$  or by  $B(k)^{-1}$ , on the right or on the left.*

(a) If  $k \in \mathbb{Z}$ ,  $k > 0$ , and  $M$  belongs to the monoid generated by  $A(k)$  and  $B(k)$ , then it has the form  $\begin{pmatrix} 1+k^2n_1 & kn_2 \\ kn_3 & 1+k^2n_4 \end{pmatrix}$  for some nonnegative integers  $n_i$ .

If  $k \in \mathbb{R}$ ,  $k > 0$ , and  $M$  belongs to the monoid generated by  $A(k)$  and  $B(k)$ , then it has the form  $\begin{pmatrix} 1+\sum_i k^i n_i & \sum_j k^j n_j \\ \sum_r k^r n_r & 1+\sum_s k^s n_s \end{pmatrix}$  where all  $n_i$  are nonnegative integers and all exponents on  $k$  are positive integers.

(b) Let  $k \in \mathbb{Z}$ ,  $k \geq 2$ . If  $M$  is a matrix from  $SL_2(\mathbb{Z})$  with nonnegative entries and no elementary operation reduces  $\sum_{i,j} m_{ij}$ , then either  $M$  is the identity matrix or  $M$  does not belong to the monoid generated by  $A(k)$  and  $B(k)$ .

Thus, for example, the matrix  $\begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$  does not belong to the monoid generated by  $A(2)$  and  $B(2)$ , although it does belong to the group generated by  $A(2)$  and  $B(2)$  by Sanov's Theorem 2.

Theorem 4 yields a simple algorithm for the membership problem in the subgroup generated by  $A(k)$  and  $B(k)$  in case  $k \in \mathbb{Z}$ ,  $k \geq 2$ . We note in passing that in general, the subgroup membership problem for  $SL_2(\mathbb{Q})$  is open, while in  $SL_2(\mathbb{Z})$  it is solvable since  $SL_2(\mathbb{Z})$  is virtually free. The general solution, based on the automatic structure of  $SL_2(\mathbb{Z})$  (see [4]), is not so transparent and has quadratic time complexity (with respect to the word length of an input). For our special subgroups we have:

**Corollary 3.** Let  $k \in \mathbb{Z}$ ,  $k \geq 2$ , and let the complexity  $|M|$  of a matrix  $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$  be the sum of all  $|m_{ij}|$ . There is an algorithm that decides whether or not a given matrix  $M \in SL_2(\mathbb{Z})$  is in the subgroup of  $SL_2(\mathbb{Z})$  generated by  $A(k)$  and  $B(k)$  (and if it does, finds a presentation of  $M$  as a group word in  $A(k)$  and  $B(k)$ ) in time  $O(n \cdot \log n)$ , where  $n = |M|$ .

**Remark.** The relation between  $|M| = \sum |m_{ij}|$  and the word length of  $M$  (with respect to the standard generators  $A(1)$  and  $B(1)$ , say) is not at all obvious and is an interesting problem in its own right.

Statement similar to Corollary 3 holds also for the monoid generated by  $A(k)$  and  $B(k)$ , for any  $k \in \mathbb{Z}$ ,  $k \geq 2$ .

The  $O(n \cdot \log n)$  is the worst-case complexity of the algorithm referred to in Corollary 3. It would be interesting to find out what the generic-case complexity (in the sense of [7]) of this algorithm is. Proposition 1 in [3] tacitly suggests that this complexity might be, in fact, sublinear in  $n = |M|$ , which would be a really interesting result, so we ask:

**Problem 1.** Is the generic-case complexity of the algorithm claimed in Corollary 3 sublinear in  $|M|$ ?

We note that, unlike the algorithms with low generic-case complexity considered in [7], this algorithm has a good chance to have low generic-case complexity giving both "yes" and "no" answers, see our Section 5 for more details.

Finally, we note that if  $M$  is in the subgroup generated by  $A(k)$  and  $B(k)$ ,  $k \geq 2$ , then the presentation of  $M$  as a group word in  $A(k)$  and  $B(k)$  is unique since the group generated by  $A(k)$  and  $B(k)$  is known to be free for any  $k \in \mathbb{R}$ ,  $k \geq 2$ , see e.g. [12]. On the other hand, the group generated by  $A(1)$  and  $B(1)$  (i.e., the whole group  $SL_2(\mathbb{Z})$ ) is not free. This implies, in particular, that for any integer  $n \geq 1$ , the group generated by  $A(\frac{1}{n})$  and  $B(\frac{1}{n})$  is not free because it contains both matrices  $A(1)$  and  $B(1)$ . Many examples of rational  $k$ ,  $0 < k < 2$ , for which the subgroup of  $SL_2(\mathbb{Q})$  generated by  $A(k)$  and  $B(k)$  is not free were found over the years, starting with [9]; see a recent paper [6] for more references. (We can single out the paper [2] where the question of non-freeness for this subgroup was reduced to solvability of particular Diophantine equations.) In particular, it is known that for any  $k$ ,  $0 < k < 2$ , of the form  $\frac{m}{mn+1}$  or  $\frac{m+n}{mn}$ ,  $m, n \in \mathbb{Z}_+$ , the group generated by  $A(k)$  and  $B(k)$  is not free. This includes  $k = \frac{2}{3}, \frac{3}{2}, \frac{3}{7}$ , etc. Also, if the group is not free for some  $k$ , then it is not free for any  $\frac{k}{n}$ ,  $n \in \mathbb{Z}_+$ .

The following problem, however, seems to be still open:

**Problem 2.** (*Yu. Merzlyakov* [1], [10]) *For which rational  $k$ ,  $0 < k < 2$ , is the group generated by  $A(k)$  and  $B(k)$  free? More generally, for which algebraic  $k$ ,  $0 < k < 2$ , is this group free?*

To the best of our knowledge, there are no known examples of a rational  $k$ ,  $0 < k < 2$ , such that the group generated by  $A(k)$  and  $B(k)$  is free. On the other hand, since any matrix from this group has the form  $\begin{pmatrix} p_{11}(k) & p_{12}(k) \\ p_{21}(k) & p_{22}(k) \end{pmatrix}$  for some polynomials  $p_{ij}(k)$  with integer coefficients, this group is obviously free if  $k$  is transcendental. For the same reason, if  $r$  and  $s$  are algebraic numbers that are Galois conjugate over  $\mathbb{Q}$ , then the group generated by  $A(r)$  and  $B(r)$  is free if and only if the group generated by  $A(s)$  and  $B(s)$  is. For example, if  $r = 2 - \sqrt{2}$ , then  $A(r)$  and  $B(r)$  generate a free group because this  $r$  is Galois conjugate to  $s = 2 + \sqrt{2} > 2$ . More generally,  $A(r)$  and  $B(r)$  generate a free group for  $r = m - n\sqrt{2}$ , and therefore also for  $r = k \cdot (m - n\sqrt{2})$ , with arbitrary positive  $k, m, n \in \mathbb{Z}$ . This implies, in particular, that the set of algebraic  $r$  for which the group is free is dense in  $\mathbb{R}$  because  $(m - n\sqrt{2})$  can be arbitrarily close to 0. All these  $r$  are irrational though.

### 3. PEAK REDUCTION

Here we prove Theorem 4 from Section 2. The method we use is called *peak reduction* and goes back to Whitehead [14], see also [8]. The idea is as follows. Given an algorithmic problem that has to be solved, one first somehow defines *complexity* of possible inputs. Another ingredient is a collection of *elementary operations* that can be applied to inputs. Thus, we now have an action of the semi-group of elementary operations on the set of inputs. Usually, of particular interest are elements of minimum complexity in any given orbit under this action. The main problem typically is to find these elements of minimum complexity. This is where the peak reduction method can be helpful. A crucial observation is: if there is a sequence of elementary operations (applied to a given input) such that at

some point in this sequence the complexity goes up (or remains unchanged) before eventually going down, then there must be a pair of *subsequent* elementary operations in this sequence (a “peak”) such that one of them increases the complexity (or leaves it unchanged), and then the other one decreases it. Then one tries to prove that such a peak can always be reduced, i.e., if there is such a pair, then there is also a single elementary operation that reduces complexity. This will then imply that there is a “greedy” sequence of elementary operations, i.e., one that reduces complexity *at every step*. This will yield an algorithm for finding an element of minimum complexity in a given orbit.

In our situation, inputs are matrices from  $SL_2(\mathbb{R})$ . For the purposes of the proof of Theorem 4, we define complexity of a matrix  $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$  to be the maximum of all  $|m_{ij}|$ . Between two matrices with the same  $\max |m_{ij}|$ , the one with the larger  $\sum_{i,j} |m_{ij}|$  has higher complexity. We will see, however, that in case of  $2 \times 2$  matrices with determinant 1, the “greedy” sequence of elementary operations would be the same as if we defined the complexity to be just  $\sum_{i,j} |m_{ij}|$ .

Elementary operations in our situation are multiplications of a matrix by either  $A(k)^{\pm 1}$  or by  $B(k)^{\pm 1}$ , on the right or on the left. They correspond to elementary row or column operations; specifically, to operations of the form  $(row_1 \pm k \cdot row_2)$ ,  $(row_2 \pm k \cdot row_1)$ ,  $(column_1 \pm k \cdot column_2)$ , and  $(column_2 \pm k \cdot column_1)$ .

We now get to

**Proof of Theorem 4.** Part (a) is established by an obvious induction on the length of a group word representing a given element of the subgroup generated by  $A(k)$  and  $B(k)$ . Part (c) follows from part (b). We omit the details and proceed to part (b).

We are going to consider various pairs of subsequent elementary operations of the following kind: the first operation increases the maximum of  $|m_{ij}|$  (or leaves it unchanged), and then the second one reduces it. We are assuming that the second elementary operation is not the inverse of the first one.

In each case like that, we show that either the maximum of  $|m_{ij}|$  in the given matrix could have been reduced by just a single elementary operation (and then  $\sum |m_{ij}|$  should be reduced, too, to keep the determinant unchanged), or  $\sum |m_{ij}|$  could have been reduced by a single elementary operation leaving the maximum of  $|m_{ij}|$  unchanged. Because of a “symmetry”, it is sufficient to consider the following cases.

First of all, we note that since the determinant of  $M$  is equal to 1, there can be 0, 2, or 4 negative entries in  $M$ . If there are 2 negative entries, they can occur either in the same row, or in the same column, or on the same diagonal. Because of the symmetry, we only consider the case where two negative entries are in the first column and the case where they are on the main diagonal. Also, cases with 0 and 4 negative entries are symmetric, so we only consider the case where there are no negative entries.

It is also convenient for us to single out the case where  $M$  has two zero entries, so we start with

**Case 0.** There are two zero entries in  $M$ . Since the determinant of  $M$  is 1, the two nonzero entries should be on a diagonal and their product should be  $\pm 1$ . If they are not on the main diagonal, then  $M^4 = I$ , in which case  $M$  cannot belong to the subgroup generated by  $A(k)$  and  $B(k)$ ,  $k \geq 2$ , since this subgroup is free.

Now suppose that the two nonzero entries are on the main diagonal, so  $M = \begin{pmatrix} x & 0 \\ 0 & \frac{1}{x} \end{pmatrix}$  for some  $x \in \mathbb{R}$ ,  $x \neq 1$ . Without loss of generality, assume  $x > 0$ . We are going to show, by way of contradiction, that such a matrix is not in the subgroup generated by  $A(k)$  and  $B(k)$ . We have:  $MA(k)M^{-1} = \begin{pmatrix} 1 & x^2k \\ 0 & 1 \end{pmatrix}$ , and

for some  $r \in \mathbb{Z}_+$  we have  $A(k)^{-r}MA(k)M^{-1} = \begin{pmatrix} 1 & yk \\ 0 & 1 \end{pmatrix}$ , where  $0 < yk \leq k$ .

If  $yk = k$ , then we have a relation  $A(k)^{-r}MA(k)M^{-1} = A(k)$ , so again we have a contradiction with the fact that the subgroup generated by  $A(k)$  and  $B(k)$  is free.

Now let  $0 < yk < k$ , so  $0 < y < 1$ , and let  $C = \begin{pmatrix} 1 & yk \\ 0 & 1 \end{pmatrix}$ . We claim that  $C$  does not belong to the subgroup generated by  $A(k)$  and  $B(k)$ . If it did, then so would the matrix  $T(m, n) = A(k)^{-m}C^n = \begin{pmatrix} 1 & (ny - m)k \\ 0 & 1 \end{pmatrix}$  for any  $m, n \in \mathbb{Z}_+$ . Since  $(ny - m)k$  can be arbitrarily close to 0, the matrix  $T(m, n)$  can be arbitrarily close to the identity matrix, which contradicts the well-known fact (see e.g. [5]) that the group generated by  $A(k)$  and  $B(k)$  is *discrete* for any  $k \geq 2$ .

In what follows, we assume that all matrices under consideration have at most one zero entry. Even though we use strict inequalities for all entries of a matrix, the reader should keep in mind that one of the inequalities may be not strict; this does not affect the argument.

**Case 1.** There are 2 negative entries, both in the first column. Thus,  $m_{11} < 0, m_{21} < 0, m_{12} > 0, m_{22} > 0$ .

**Case 1a.** Two subsequent elementary operations reducing some entry after increasing it are both  $(column_1 + k \cdot column_2)$ . If, after one operation  $(column_1 + k \cdot column_2)$ , the element  $m_{11}$ , say, becomes positive, then  $m_{21}$  should become positive, too, for the determinant to remain unchanged. Then, after applying  $(column_1 + k \cdot column_2)$  one more time, new  $|m_{11}|$  will become greater than it was, contrary to the assumption.

If, after one operation  $(column_1 + k \cdot column_2)$ ,  $m_{11}$  remains negative, then this operation reduces  $|m_{11}|$ , and this same operation should also reduce  $|m_{21}|$  for the determinant to remain unchanged. Indeed, the determinant is  $m_{11}m_{22} - m_{12}m_{21} = 1$ . If  $|m_{11}|$  decreases while  $m_{11}$  remains negative, then the value of  $m_{11}m_{22}$  increases (but remains negative). Therefore, the value of  $m_{12}m_{21}$  should increase, too, for the difference to remain unchanged. Since  $m_{21}$  should remain negative, this implies that  $|m_{21}|$  should decrease, hence  $\sum |m_{ij}|$  decreases after one operation  $(column_1 + k \cdot column_2)$ .

The same kind of argument works in the case where both operations are  $(row_1 - k \cdot row_2)$ ,

If both operations are  $(row_1 + k \cdot row_2)$ , or  $(column_1 - k \cdot column_2)$ , then no  $|m_{ij}|$  can possibly decrease since  $m_{11} < 0, m_{21} < 0, m_{12} > 0, m_{22} > 0$ .

**Case 1b.** Two subsequent elementary operations are:  $(row_1 + k \cdot row_2)$ , followed by  $(column_1 + k \cdot column_2)$ . The result of applying these two operations to the matrix  $M$  is: 
$$\begin{pmatrix} m_{11} + k \cdot m_{21} + k \cdot m_{12} + k^2 m_{22} & m_{12} + k \cdot m_{22} \\ m_{21} + k \cdot m_{22} & m_{22} \end{pmatrix}$$
. This case is nontrivial only if the first operation increases the absolute value of the element in the top left corner (or leaves it unchanged), and then the second operation reduces it. But then the second operation should also reduce the absolute value of  $m_{21}$  for the determinant to remain unchanged. This means a single operation  $(column_1 + k \cdot column_2)$  would reduce  $|m_{21}|$  to begin with, and this same operation should also reduce  $|m_{11}|$  for the determinant to remain unchanged. Thus, a single elementary operation would reduce the complexity of  $M$ .

The same argument takes care of any of the following pairs of subsequent elementary operations:  $(row_1 \pm k \cdot row_2)$ , followed by  $(column_1 \pm k \cdot column_2)$ , as well as  $(row_1 \pm k \cdot row_2)$ , followed by  $(column_2 \pm k \cdot column_1)$ .

**Case 1c.** Two subsequent elementary operations are:  $(row_1 + k \cdot row_2)$ , followed by  $(row_2 + k \cdot row_1)$ . The result of applying these two operations to the matrix  $M$  is: 
$$\begin{pmatrix} m_{11} + k \cdot m_{21} & m_{12} + k \cdot m_{22} \\ (k^2 + 1)m_{21} + k \cdot m_{11} & (k^2 + 1)m_{22} + k \cdot m_{12} \end{pmatrix}$$
. In this case, obviously  $|(k^2 + 1)m_{21} + k \cdot m_{11}| > |m_{21}|$  and  $|(k^2 + 1)m_{22} + k \cdot m_{12}| > |m_{22}|$ , so we do not have a decrease, i.e., this case is moot.

**Case 1d.** Two subsequent elementary operations are:  $(row_1 + k \cdot row_2)$ , followed by  $(row_2 - k \cdot row_1)$ . The result of applying these two operations to the matrix  $M$  is: 
$$\begin{pmatrix} m_{11} + k \cdot m_{21} & m_{12} + k \cdot m_{22} \\ -(k^2 - 1)m_{21} - k \cdot m_{11} & -(k^2 - 1)m_{22} - k \cdot m_{12} \end{pmatrix}$$
. If  $k > \sqrt{2}$ , then  $|-(k^2 - 1)m_{21} - k \cdot m_{11}| > |m_{21}|$  and  $|-(k^2 - 1)m_{22} - k \cdot m_{12}| > |m_{22}|$ , so we do not have a decrease, i.e., this case is moot, too.

**Case 1e.** Two subsequent elementary operations are:  $(row_1 - k \cdot row_2)$ , followed by  $(row_2 + k \cdot row_1)$ . The result of applying these two operations to the matrix  $M$  is: 
$$\begin{pmatrix} m_{11} - k \cdot m_{21} & m_{12} - k \cdot m_{22} \\ k \cdot m_{11} - (k^2 - 1) \cdot m_{21} & k \cdot m_{12} - (k^2 - 1) \cdot m_{22} \end{pmatrix}$$
. If the first operation increases  $|M|$  (or leaves it unchanged) and then the second one reduces it, then the second operation should reduce  $|m_{21}|$  or  $|m_{22}|$ . Assume, without loss of generality, that  $|m_{21}| \geq |m_{22}|$ .

We may assume that  $|m_{11} - k \cdot m_{21}| \geq |m_{11}| = -m_{11}$  and  $|m_{12} - k \cdot m_{22}| \geq |m_{12}| = m_{12}$  because otherwise, the complexity of  $M$  could be reduced by a single operation  $(row_1 - k \cdot row_2)$ .

Now we look at the inequality  $|k \cdot m_{11} - (k^2 - 1) \cdot m_{21}| < |m_{21}| = -m_{21}$ . Rewrite it as follows:  $|k \cdot (m_{11} - k \cdot m_{21}) + m_{21}| < -m_{21}$ . We may assume that  $m_{11} - k \cdot m_{21} > 0$  because otherwise, a single operation  $(row_1 - k \cdot row_2)$  would reduce the complexity of  $M$ . We also know that  $|m_{11} - k \cdot m_{21}| \geq -m_{11}$  (see the previous paragraph). Thus,  $m_{11} - k \cdot m_{21} \geq -m_{11}$ . This inequality, together with  $|k \cdot (m_{11} - k \cdot$

$m_{21}) + m_{21}| < -m_{21}$ , yield  $|-k \cdot m_{11} + m_{21}| < -m_{21}$ . This means a single operation  $(row_2 - k \cdot row_1)$  would reduce  $|m_{21}|$ .

**Case 1f.** Two subsequent elementary operations are:  $(row_1 - k \cdot row_2)$ , followed by  $(row_2 - k \cdot row_1)$ . The result of applying these two operations to the matrix  $M$  is:  $\begin{pmatrix} m_{11} - k \cdot m_{21} & m_{12} - k \cdot m_{22} \\ (k^2 + 1) \cdot m_{21} - k \cdot m_{11} & (k^2 + 1) \cdot m_{22} - k \cdot m_{12} \end{pmatrix}$ . If the first operation increases  $|M|$  (or leaves it unchanged) and then the second one reduces it, then the second operation should reduce  $|m_{21}|$  or  $|m_{22}|$ .

We may assume that  $m_{12} - k \cdot m_{22} < 0$  because otherwise, a single operation  $(row_1 - k \cdot row_2)$  would reduce  $|m_{12}|$  and therefore also  $|m_{11}|$  for the determinant to remain unchanged in this case. Now look at the element in the bottom right corner:  $(k^2 + 1) \cdot m_{22} - k \cdot m_{12} = k \cdot (k \cdot m_{22} - m_{12}) + m_{22}$ . Since  $k \cdot m_{22} - m_{12} > 0$ , we have  $|(k^2 + 1) \cdot m_{22} - k \cdot m_{12}| > |m_{22}|$ , a contradiction.

**Case 1g.** Two subsequent elementary operations are:  $(column_1 + k \cdot column_2)$ , followed by  $(column_2 + k \cdot column_1)$ . The result of applying these two operations to the matrix  $M$  is:  $\begin{pmatrix} m_{11} + k \cdot m_{12} & (k^2 + 1) \cdot m_{12} + k \cdot m_{11} \\ m_{21} + k \cdot m_{22} & (k^2 + 1) \cdot m_{22} + k \cdot m_{21} \end{pmatrix}$ . If the first operation increases  $|M|$  (or leaves it unchanged) and then the second one reduces it, then the second operation should reduce  $|m_{12}|$  or  $|m_{22}|$ . Let us assume here that  $|m_{22}| \geq |m_{12}|$ .

Now look at the element in the bottom right corner:  $(k^2 + 1) \cdot m_{22} + k \cdot m_{21} = k \cdot (m_{21} + k \cdot m_{22}) + m_{22}$ . We may assume that  $m_{21} + k \cdot m_{22} > 0$  because otherwise, a single operation  $(column_1 + k \cdot column_2)$  would reduce  $|m_{21}|$ . In that case, however, we have  $|(k^2 + 1) \cdot m_{22} + k \cdot m_{21}| = k \cdot (m_{21} + k \cdot m_{22}) + m_{22} > m_{22} = |m_{22}|$ , a contradiction.

**Case 1h.** Two subsequent elementary operations are:  $(column_1 + k \cdot column_2)$ , followed by  $(column_2 - k \cdot column_1)$ . The result of applying these two operations to the matrix  $M$  is:  $\begin{pmatrix} m_{11} + k \cdot m_{12} & (1 - k^2) \cdot m_{12} - k \cdot m_{11} \\ m_{21} + k \cdot m_{22} & (1 - k^2) \cdot m_{22} - k \cdot m_{21} \end{pmatrix}$ . If the first operation increases  $|M|$  (or leaves it unchanged) and then the second one reduces it, then the second operation should reduce  $|m_{12}|$  or  $|m_{22}|$ . Assume here that  $|m_{22}| \geq |m_{12}|$ . Then we should have  $|(1 - k^2) \cdot m_{22} - k \cdot m_{21}| < |m_{22}| = m_{22}$ .

We may assume that  $m_{21} + k \cdot m_{22} > 0$  because otherwise, a single operation  $(column_1 + k \cdot column_2)$  would reduce  $|m_{21}|$  while keeping the element in this position negative. Then this same operation should reduce  $|m_{11}|$ , too, for the determinant to remain unchanged. Also, since the first operation was supposed to increase  $|M|$  (or leave it unchanged), we should have, in particular,  $|m_{21} + k \cdot m_{22}| = m_{21} + k \cdot m_{22} \geq |m_{21}| = -m_{21}$ . This, together with the inequality in the previous paragraph, gives  $|(1 - k^2) \cdot m_{22} - k \cdot m_{21}| = |m_{22} - k \cdot (m_{21} + k \cdot m_{22})| \geq |k \cdot m_{22} + m_{21}|$ . Therefore, we should have  $|k \cdot m_{22} + m_{21}| < |m_{22}| = m_{22}$ . Since  $k \geq 2$ , this implies  $|m_{21}| > |m_{22}|$ , contradicting the assumption of  $m_{22}$  having the maximum absolute value in the matrix  $M$ .

**Case 1i.** Two subsequent elementary operations are:  $(column_1 - k \cdot column_2)$ , followed by  $(column_2 + k \cdot column_1)$ . The result of applying these two operations

to the matrix  $M$  is:  $\begin{pmatrix} m_{11} - k \cdot m_{12} & (1 - k^2) \cdot m_{12} + k \cdot m_{11} \\ m_{21} - k \cdot m_{22} & (1 - k^2) \cdot m_{22} + k \cdot m_{21} \end{pmatrix}$ . Since  $k \geq 2$ , we have  $|(1 - k^2) \cdot m_{22} + k \cdot m_{21}| > |m_{22}|$ , so this case is moot.

**Case 1j.** Two subsequent elementary operations are:  $(\text{column}_1 - k \cdot \text{column}_2)$ , followed by  $(\text{column}_2 - k \cdot \text{column}_1)$ . The result of applying these two operations to the matrix  $M$  is:  $\begin{pmatrix} m_{11} - k \cdot m_{12} & (1 + k^2) \cdot m_{12} - k \cdot m_{11} \\ m_{21} - k \cdot m_{22} & (1 + k^2) \cdot m_{22} - k \cdot m_{21} \end{pmatrix}$ . Since  $|(1 + k^2) \cdot m_{22} - k \cdot m_{21}| = |m_{22} - k \cdot (m_{21} - k \cdot m_{22})| > |m_{22}|$ , this case is moot, too.

**Case 2.** Two negative entries are on a diagonal. Without loss of generality, we assume here that  $m_{11} > 0, m_{22} > 0, m_{12} < 0, m_{21} < 0$ . Because of the ‘‘symmetry’’ between row and column operations in this case, we can reduce the number of subcases (compared to Case 1 above) and only consider the following.

**Case 2a.** Two subsequent elementary operations are:  $(\text{row}_1 + k \cdot \text{row}_2)$ , followed by  $(\text{column}_1 + k \cdot \text{column}_2)$ . The result of applying these two operations to the matrix  $M$  is:  $\begin{pmatrix} m_{11} + k \cdot m_{21} + k \cdot m_{12} + k^2 m_{22} & m_{12} + k \cdot m_{22} \\ m_{21} + k \cdot m_{22} & m_{22} \end{pmatrix}$ . This case is nontrivial only if the first operation increases the absolute value of the element in the top left corner (or leaves it unchanged), and then the second operation reduces it. First let us look at the element  $m_{21} + k \cdot m_{22}$ . If  $m_{21} + k \cdot m_{22} < 0$ , then a single operation  $(\text{column}_1 + k \cdot \text{column}_2)$  would reduce  $m_{21}$  while keeping that element negative. In that case, this operation would reduce  $m_{11}$ , too, while keeping it positive because otherwise, the determinant would change. Thus, a single operation  $(\text{column}_1 + k \cdot \text{column}_2)$  would reduce the complexity of  $M$  in that case. The same argument shows that  $m_{12} + k \cdot m_{22} \geq 0$  because otherwise, a single operation  $(\text{row}_1 + k \cdot \text{row}_2)$  would reduce the complexity of  $M$ .

If  $m_{21} + k \cdot m_{22} \geq 0$  and  $m_{12} + k \cdot m_{22} \geq 0$ , then  $m_{11} + k \cdot m_{21} + k \cdot m_{12} + k^2 m_{22} \geq 0$  for the determinant to be equal to 1. But then the second operation should also reduce the absolute value of  $m_{21}$  for the determinant to remain unchanged. This means a single operation  $(\text{column}_1 + k \cdot \text{column}_2)$  would reduce  $|m_{21}|$  to begin with, and this same operation should also reduce  $|m_{11}|$  for the determinant to remain unchanged, so this single operation would reduce the complexity of  $M$ .

The same argument takes care of any of the following pairs of subsequent elementary operations:  $(\text{row}_1 \pm k \cdot \text{row}_2)$ , followed by  $(\text{column}_1 \pm k \cdot \text{column}_2)$ , as well as  $(\text{row}_1 \pm k \cdot \text{row}_2)$ , followed by  $(\text{column}_2 \pm k \cdot \text{column}_1)$ .

**Case 2b.** Two subsequent elementary operations are:  $(\text{row}_1 + k \cdot \text{row}_2)$ , followed by  $(\text{row}_2 + k \cdot \text{row}_1)$ . The result of applying these two operations to the matrix  $M$  is:  $\begin{pmatrix} m_{11} + k \cdot m_{21} & m_{12} + k \cdot m_{22} \\ (k^2 + 1)m_{21} + k \cdot m_{11} & (k^2 + 1)m_{22} + k \cdot m_{12} \end{pmatrix}$ . If the first operation increases  $|M|$  (or leaves it unchanged) and then the second one reduces it, then the second operation should reduce  $|m_{21}|$  or  $|m_{22}|$ . Assume, without loss of generality, that  $|m_{21}| \geq |m_{22}|$ .

Thus, we have  $|(k^2 + 1)m_{21} + k \cdot m_{11}| < |m_{21}| = -m_{21}$ . At the same time, we may assume that  $|m_{11} + k \cdot m_{21}| \geq |m_{11}| = m_{11}$  because otherwise, a single operation  $(\text{row}_1 + k \cdot \text{row}_2)$  would reduce the complexity of  $M$ .

Then, if  $m_{11} + k \cdot m_{21} > 0$ , then a single operation  $(row_1 + k \cdot row_2)$  would reduce  $|m_{11}|$ , and therefore also  $|m_{12}|$ . Thus, we may assume that  $m_{11} + k \cdot m_{21} < 0$ .

Now let us look at the inequality  $|(k^2 + 1)m_{21} + k \cdot m_{11}| < -m_{21}$ . We know that  $m_{11} + k \cdot m_{21} < 0$ . Therefore,  $k \cdot m_{11} + k^2 \cdot m_{21} < 0$ . Now  $|(k^2 + 1)m_{21} + k \cdot m_{11}| = |k \cdot m_{11} + k^2 \cdot m_{21} + m_{21}| > -m_{21}$ . This contradiction completes Case 2b.

**Case 2c.** Two subsequent elementary operations are:  $(row_1 + k \cdot row_2)$ , followed by  $(row_2 - k \cdot row_1)$ . The result of applying these two operations to the matrix  $M$  is:

$\begin{pmatrix} m_{11} + k \cdot m_{21} & m_{12} + k \cdot m_{22} \\ -(k^2 - 1)m_{21} - k \cdot m_{11} & -(k^2 - 1)m_{22} - k \cdot m_{12} \end{pmatrix}$ . The analysis here is similar to the previous Case 2b. First we note that we may assume  $m_{11} + k \cdot m_{21} < 0$ , so  $-k \cdot m_{11} - k^2 \cdot m_{21} > 0$ . On the other hand, we may assume that  $|m_{11} + k \cdot m_{21}| = -m_{11} - k \cdot m_{21} \geq m_{11}$  because otherwise, a single operation  $(row_1 + k \cdot row_2)$  would reduce  $|m_{11}|$ . Therefore,  $-k \cdot m_{11} - k^2 \cdot m_{21} \geq k \cdot m_{11}$ .

This, together with the inequality  $|-(k^2 - 1)m_{21} - k \cdot m_{11}| = |-k \cdot m_{11} - k^2 \cdot m_{21} + m_{21}| < |m_{21}| = -m_{21}$ , implies  $|k \cdot m_{11} + m_{21}| < |m_{21}|$ , in which case a single operation  $(row_2 + k \cdot row_1)$  would reduce  $|m_{21}|$ .

**Case 2d.** Two subsequent elementary operations are:  $(row_1 - k \cdot row_2)$ , followed by  $(row_2 + k \cdot row_1)$ . The result of applying these two operations to the matrix

$M$  is:  $\begin{pmatrix} m_{11} - k \cdot m_{21} & m_{12} - k \cdot m_{22} \\ -(k^2 - 1)m_{21} + k \cdot m_{11} & -(k^2 - 1)m_{22} + k \cdot m_{12} \end{pmatrix}$ . If  $k^2 > 2$ , here we obviously have  $|-(k^2 - 1)m_{22} + k \cdot m_{12}| > |m_{22}|$ , and  $|-(k^2 - 1)m_{21} + k \cdot m_{11}| > |m_{21}|$ . Thus, this case is moot.

**Case 2e.** Two subsequent elementary operations are:  $(row_1 - k \cdot row_2)$ , followed by  $(row_2 - k \cdot row_1)$ . The result of applying these two operations to the matrix  $M$  is:

$\begin{pmatrix} m_{11} - k \cdot m_{21} & m_{12} - k \cdot m_{22} \\ (k^2 + 1)m_{21} - k \cdot m_{11} & (k^2 + 1)m_{22} - k \cdot m_{12} \end{pmatrix}$ . Here again we obviously have  $|(k^2 + 1)m_{22} - k \cdot m_{12}| > |m_{22}|$  and  $|(k^2 + 1)m_{21} - k \cdot m_{11}| > |m_{21}|$ , so this case is moot, too.

**Case 3.** There are no negative entries. Because of the obvious symmetry, it is sufficient to consider the following cases.

**Case 3a.** Two subsequent elementary operations are both  $(column_1 + k \cdot column_2)$ ,  $(column_2 + k \cdot column_1)$ ,  $(row_1 + k \cdot row_2)$  or  $(row_2 + k \cdot row_1)$ . Since all the entries are positive, the second operation cannot decrease the complexity in this case.

**Case 3b.** Two subsequent elementary operations are  $(column_1 + k \cdot column_2)$ , followed by  $(column_2 - k \cdot column_1)$ , give the matrix

$\begin{pmatrix} m_{11} + k \cdot m_{12} & (1 - k^2) \cdot m_{12} - k \cdot m_{11} \\ m_{21} + k \cdot m_{22} & (1 - k^2) \cdot m_{22} - k \cdot m_{21} \end{pmatrix}$ . If the first operation increases  $|M|$  (or leaves it unchanged) and then the second operation reduces it, then the second operation should reduce, say,  $|m_{12}| = m_{12}$  (assuming that  $m_{12} \geq m_{22}$ ).

After the first operation we note that  $|m_{11} + k \cdot m_{12}| = m_{11} + k \cdot m_{12} \geq |m_{12}| = m_{12}$ . After the second operation, assuming that complexity was reduced, we have  $|(1 - k^2)m_{12} - k \cdot m_{11}| \leq |m_{12}| = m_{12}$ , which can be rewritten as  $|m_{12} - k(m_{11} + k \cdot m_{12})| \leq m_{12}$ . Since all the entries are positive, we have  $m_{11} + k \cdot m_{12} \geq m_{12}$ , and

hence in order for the second operation to reduce  $|m_{12}|$ , the following inequality should hold:  $-m_{12} \leq (1 - k^2)m_{12} - k \cdot m_{11} \leq 0$ . Subtracting  $m_{12}$ , multiplying each part by  $-1$  and factoring out  $k$  we get  $2m_{12} \geq k(m_{11} + m_{12}) \geq m_{12}$ . Divide by  $k$ :  $\frac{2}{k} \cdot m_{12} \geq m_{11} + k \cdot m_{12} \geq \frac{1}{k} \cdot m_{12}$ . However,  $k \geq 2$  implies that  $m_{11} + k \cdot m_{12} \leq m_{12}$ , which brings us to a contradiction.

**Case 3c.** Two subsequent elementary operations are  $(\text{column}_1 - k \cdot \text{column}_2)$ , followed by  $(\text{column}_2 + k \cdot \text{column}_1)$ . The resulting matrix is

$$\begin{pmatrix} m_{11} - k \cdot m_{12} & (1 - k^2) \cdot m_{12} + k \cdot m_{11} \\ m_{21} - k \cdot m_{22} & (1 - k^2) \cdot m_{22} + k \cdot m_{21} \end{pmatrix}. \text{ If the first operation increases } |M|$$

(or leaves it unchanged) and then the second one reduces it, then the second operation should reduce, say,  $|m_{12}| = m_{12}$  (assuming that  $m_{12} \geq m_{22}$ ).

Also, after the first operation we can observe that  $m_{11} \leq |m_{11} - k \cdot m_{12}| =$  and  $m_{21} \leq |m_{21} - k \cdot m_{22}|$  because otherwise, a single operation  $(\text{column}_1 - k \cdot \text{column}_2)$  would reduce  $|m_{11}|$  and  $|m_{21}|$ . This implies that  $m_{11} - k \cdot m_{12} \leq -m_{11}$  and  $m_{21} - k \cdot m_{22} \leq -m_{21}$ .

Consider the inequality  $|(1 - k^2)m_{12} + k \cdot m_{11}| \leq m_{12}$ . Rewrite it in the following way:  $|m_{12} - k \cdot (k \cdot m_{12} - m_{11})| \leq m_{12}$ . Combining this with the previous inequalities we get  $|m_{12} - k \cdot m_{11}| \leq |m_{12} - k \cdot (k \cdot m_{12} - m_{11})| \leq m_{12}$ . Therefore, a single operation  $(\text{column}_2 - k \cdot \text{column}_1)$  reduces  $|m_{12}|$ .

**Case 3d.** Two subsequent elementary operations are  $(\text{column}_1 - k \cdot \text{column}_2)$  followed by  $(\text{column}_2 - k \cdot \text{column}_1)$ . The resulting matrix is

$$\begin{pmatrix} m_{11} - k \cdot m_{12} & (1 - k^2) \cdot m_{12} - k \cdot m_{11} \\ m_{21} - k \cdot m_{22} & (1 - k^2) \cdot m_{22} - k \cdot m_{21} \end{pmatrix}. \text{ If the first operation increases } |M|$$

(or leaves it unchanged) and then the second operation reduces it, then the second one should reduce, say,  $|m_{12}| = m_{12}$  (assuming that  $m_{12} \geq m_{22}$ ).

We may assume that  $m_{11} - k \cdot m_{12} \leq 0$  because if  $m_{11} - k \cdot m_{12} > 0$ , then for the determinant to be unchanged after the first operation, we would have also  $m_{21} - k \cdot m_{22} > 0$ , but then a single operation  $(\text{column}_1 - k \cdot \text{column}_2)$  would reduce the complexity of  $M$ .

Thus,  $|m_{11} - k \cdot m_{12}| = k \cdot m_{12} - m_{11}$ . If the first operation  $(\text{column}_1 - k \cdot \text{column}_2)$  did not reduce the complexity of  $M$ , then  $m_{11} \leq k \cdot m_{12} - m_{11}$ , which implies that  $m_{11} - k \cdot m_{12} \leq -m_{11}$ .

Now consider the inequality  $|(1 - k^2)m_{12} - k \cdot m_{11}| \leq m_{12}$ . After rewriting it we get  $|m_{12} - k \cdot (k \cdot m_{12} - m_{11})| \leq m_{12}$ . Combining it with the inequality in the previous paragraph, we get  $|m_{12} - k \cdot m_{11}| \leq |m_{12} - k \cdot (k \cdot m_{12} - m_{11})| \leq m_{12}$ . Therefore, a single elementary operation  $(\text{column}_2 - k \cdot \text{column}_1)$  reduces  $|m_{12}|$ .

#### 4. PROOF OF THEOREM 3

Let  $k, m \in \mathbb{Z}$ ,  $k \geq 3$ ,  $m \geq 1$ . Denote  $M(k, m) = \begin{pmatrix} 1 - k^2 m & k^2 m \\ -k^2 m & 1 + k^2 m \end{pmatrix}$ . It is straightforward to check that:

- (1)  $M(k, m)$  has determinant 1;
- (2)  $M(k, m) = M(k, 1)^m$ ;

(3) No elementary  $k$ -operation reduces the absolute value of any of the entries of  $M(k, m)$ .

Since the cyclic group generated by  $M(k, 1)$  is infinite, the result follows from Theorem 4.

## 5. PROOF OF COROLLARY 3

We assume in this section that  $k \geq 3$  because for  $k = 2$ , the membership problem in the subgroup of  $SL_2(\mathbb{Z})$  generated by  $A(k)$  and  $B(k)$  is solvable in constant time, see Corollary 2 in the Introduction.

First of all we check that a given matrix  $M$  from the group  $SL_2(\mathbb{Z})$  has the form  $\begin{pmatrix} 1 + k^2 n_1 & kn_2 \\ kn_3 & 1 + k^2 n_4 \end{pmatrix}$  for some integers  $n_i$ . Then we check that  $M$  has at most one zero entry. If there are more, then  $M$  does not belong to the subgroup in question unless  $M$  is the identity matrix. We also check that  $\max |m_{ij}| > 1$ . If  $\max |m_{ij}| = 1$ , then  $M$  does not belong to the subgroup in question unless  $M$  is the identity matrix. Indeed, the only nontrivial cases here are  $M = A(\pm 1)$  and  $M = B(\pm 1)$ . Then  $M^k = A(k)^{\pm 1}$  or  $M^k = B(k)^{\pm 1}$ . This would give a nontrivial relation in the group generated by  $A(k)$  and  $B(k)$  contradicting the fact that this group is free.

Now let  $\max |m_{ij}| > 1$ . If no elementary operation either reduces  $\max |m_{ij}|$  or reduces  $\sum_{i,j} |m_{ij}|$  without increasing  $\max |m_{ij}|$ , then  $M$  does not belong to the subgroup generated by  $A(k)$  and  $B(k)$ . If there is an elementary operation that reduces  $\max |m_{ij}|$ , then we apply it. For example, suppose the elementary operation ( $\text{row}_1 - k \cdot \text{row}_2$ ) reduces  $|m_{11}|$ . The result of this operation is the matrix  $\begin{pmatrix} m_{11} - km_{21} & m_{12} - km_{22} \\ m_{21} & m_{22} \end{pmatrix}$ . If  $|m_{11}|$  here has decreased, then  $|m_{12}|$  could not increase because otherwise, the determinant of the new matrix would not be equal to 1. Thus, the complexity of the matrix  $M$  has been reduced, and the new matrix belongs to our subgroup if and only if the matrix  $M$  does. Since there are only finitely many numbers of the form  $kn, n \in \mathbb{Z}$ , with bounded absolute value, this process should terminate either with a non-identity matrix whose complexity cannot be reduced or with the identity matrix. In the latter case, the given matrix was in the subgroup generated by  $A(k)$  and  $B(k)$ ; in the former case, it was not.

To estimate the time complexity of this algorithm, we note that each step of it (i.e., applying a single elementary operation) takes time  $O(\log m)$ , where  $m$  is the complexity of the matrix this elementary operation is applied to. This is because if  $k$  is an integer, multiplication by  $k$  amounts to  $k - 1$  additions, and each addition of integers not exceeding  $m$  takes time  $O(\log m)$ . Since the complexity of a matrix is reduced at least by 1 at each step of the algorithm, the total complexity is  $O(\sum_{k=1}^n \log k) = O(n \cdot \log n)$ . This completes the proof.  $\square$

As for generic-case complexity of this algorithm (cf. Problem 1 in our Section 2), we note that, speaking very informally, a ‘‘random’’ product of  $A(k)^{\pm 1}$  and  $B(k)^{\pm 1}$  is ‘‘close’’ to a product where  $A(k)^{\pm 1}$  and  $B(k)^{\pm 1}$  alternate, in which case the complexity of the product matrix grows exponentially in the number of factors

(see e.g. [3, Proposition 1]), so the number of summands in the sum that appears in the proof of Corollary 3 will be logarithmic in  $n$ , and therefore generic-case complexity of the algorithm should be  $O(\log^2 n)$  in case the answer is “yes” (i.e., an input matrix belongs to the subgroup generated by  $A(k)$  and  $B(k)$ ). Of course, a “random” matrix from  $SL_2(\mathbb{Z})$  will *not* belong to the subgroup generated by  $A(k)$  and  $B(k)$  with overwhelming probability. This is because if  $k \geq 3$ , this subgroup has infinite index in  $SL_2(\mathbb{Z})$ . It is, however, not clear how fast (generically) our algorithm will detect that; specifically, whether it will happen in sublinear time or not.

Note that, unlike the algorithms with low generic-case complexity considered in [7], this algorithm has a good chance to have low generic-case complexity giving both “yes” and “no” answers.

Finally, we note that generic-case complexity depends on how one defines the *asymptotic density* of a subset of inputs in the set of all possible inputs. This, in turn, depends on how one defines the complexity of an input. In [7], complexity of an element of a group  $G$  was defined as the minimum word length of this element with respect to a fixed generating set of  $G$ . In our situation, where inputs are matrices over  $\mathbb{Z}$ , it is probably more natural to define complexity  $|M|$  of a matrix  $M$  as the sum of the absolute values of the entries of  $M$ , like we did in this paper. Yet another natural way is to use *Kolmogorov complexity*, i.e., speaking informally, the minimum possible size of a description of  $M$ . Since Kolmogorov complexity of an integer  $n$  is equivalent to  $\log n$ , we see that for a matrix  $M \in SL_2(\mathbb{Z})$ , Kolmogorov complexity is equivalent to  $\log |M|$ , for  $|M| = \sum |m_{ij}|$ , as defined in this paper. This is not the case though if  $M \in SL_2(\mathbb{Q})$  since for Kolmogorov complexity of a rational number, complexity of both the numerator and denominator matters.

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